Outline

$$\begin{tikzcd} \text{Reps of p-adics} \arrow[r] & \text{Affine Hecke algebras} \arrow[r] & \text{Graded affine Hecke algebras} \arrow[r, "\text{Equivariant homology}"] & \text{parametrizat}^{n} \text{ of ireps.} \end{tikzcd}$$

How to get from p-adics to Hecke algebras?

Recall: Main thm. from last time.

Depth zero $V \Leftarrow\Rightarrow$ contains minimal K-type (of depth 0) $\underline{\underline{(\rho, W)}}$

↳ subrep of parahoric $P = G_{x,0}$ trivial on $U = G_{x,0^+}$ & cuspidal rep. of $P/U = G_{x,0}/G_{x,0^+}$

Unipotent rep ⟶ V

unipotent

Also, the minimal K-types satisfy certain property of being associate, so that they can 'parametise' the unipotent reps.

$P^{(=K)}$ cpt open. Then V is K-semisimple.

For the min-K-type (ρ, W), have: $e_\rho(x) := \dfrac{\dim \rho}{\mu(K)} \underbrace{\mathrm{tr}_W}_{\text{character}}(\rho(x^{-1})) \quad (x \in K)$

is idempotent in $\mathcal{H}(G)$ which is project. onto ρ-isotypic subspace.

Mp: V irred & ρ-isotypic subspace $\neq 0 \Rightarrow$ gen by it.

So: consider

$$\mathcal{C}_{e_\rho}(G) \xleftrightarrow{\text{"abstract alg."}} e_\rho \mathcal{H}(G) e_\rho\text{-mod}.$$

$\vdots$ subcat of reps V gen by $e_\rho V$ & closed under subquotients

$$\cong \mathrm{End}_G(\mathcal{H}(G)e_\rho)$$

$$\cong \mathrm{End}_G\Big(\mathcal{H}(G)e_\rho \otimes_{e_\rho \mathcal{H}(K) e_\rho} \overset{W^* \otimes_{\mathbb{C}} W^* \ \cong}{\mathrm{End}_{\mathbb{C}}(W)}\Big)$$

$$\cong \mathrm{End}_G(\text{c-Ind}_K^G \rho \otimes_{\mathbb{C}} W^*)$$

$$\cong \mathrm{End}_G(\text{c-Ind}_K^G \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}} W\text{-mod} \xleftrightarrow{\text{Morita equiv.}} \mathrm{End}_G(\text{c-Ind}_K^G \rho)\text{-mod}$$

$\mathrm{End}_G(\text{c-Ind}_K^G \rho) =: \mathcal{H}(G, \rho)$

$\cong$ (Fiona's notes) algebra of fns $f : G \to \mathrm{End}_{\mathbb{C}} \tilde{W}$ (↳ smooth dual) (w cpt supp) s.t. $f(k_1 g k_2) = \tilde{\rho}(k_1) f(g) \tilde{\rho}(k_2)$

$\mathcal{H}(G, \rho)$ is an affine Hecke algebra!

E.g. $K = P = I$, ρ = trivial rep,

recover $C_c^\infty(I \backslash G / I)$ f. last week.

(V has "Iwahori-fixed vector")

Problem is: ~~the~~ $\mathcal{H}(G, \rho)$ may have unequal parameters!

Best way to see this: already in finite gp case, $\mathrm{End}(\phi_P^{G})$ (ϕ cuspidal unipotent) has unequal parameters!

Ref: Carter p. 464 has a table of parameters.

Even in type B, C

Def^n of affine Hecke algebras.

Root datum $(X, Y, ~~Y~~, R, R^\vee, \Pi)$

Notations: W^e = extended affine Weyl gp $= X \rtimes W_0$.
W^a = affine Weyl gp $= \mathbb{Z}R \rtimes W_0$ (Coxeter gp.)
W_0 = finite Weyl gp

Recall: finite Hecke algebra is deformat^n of group alg. of W_0.
Similarly:

Def^n: (Affine Hecke algebra)
$\mathcal{H}$ = $\mathbb{C}[v, v^{-1}]$-alg. with generators (basis) $\{T_w \mid w \in W^e\}$.
modulo:

$$T_w T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w')$$
$$(T_s+1)(T_s - (v^{L(s)})^2) = 0 \quad (s \text{ simple affine reflect}^n)$$
} Iwahori presentat^n

Rmk: Length f^n ℓ for W^a: choose fundamental alcove A_0 & count no. of hyperplanes between A_0 & $w(A_0)$.
Extend to W^e in same way.

Here $L(s)$ are the parameters of $\mathcal{H}$.
Only condit^n: $L(s) = L(s')$ when s, s' conjugate in W^e.

Rmk: Also can rescale by setting $T_s = v^{L(s)} N_s$ to obtain relat^n $(N_s - v^{L(s)})(N_s + v^{-L(s)}) = 0$. (Will use T_s today.)

Of course, from $W^e = X \rtimes W_0$, expect to have "affine part" & "finite part".

Prop/Def^n: (Bernstein-Lusztig presentat^n)

$$\mathcal{H} = \underset{\text{"affine part"}}{\mathbb{C}[X]} \otimes_{\mathbb{C}} \underset{\text{"finite part"}}{\mathcal{H}(W_0)} \quad \text{(as vector spaces, say)}$$

basis $\{\theta_x \mid x \in X\}$ basis $\{T_w \mid w \in W_0\}$.

Rmk: $\theta_x = N_{t_x}$ for x dominant.

Notat^n: Write Θ for the $\mathbb{C}[v, v^{-1}]$ submod gen by θ_x and call it the affine part.

Remains to specify cross-relat^n between $\mathbb{C}[X]$ & $\mathcal{H}(W_0)$.
$s = s_\alpha \in W_0$ simple reflect^n.
If look at Lusztig's paper, split between $\alpha^\vee \in 2Y$ & $\notin 2Y$.

$$\theta_x T_s - T_s \theta_{s(x)} = \begin{cases} ((v^{L(s)})^2 - 1) \dfrac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}} & (\alpha^\vee \notin 2Y). \\ ? & (\alpha^\vee \in 2Y). \end{cases}$$

Digression on $\check{\alpha} \in 2Y$:

Key point: parameters L depend on conjugacy in W^e.

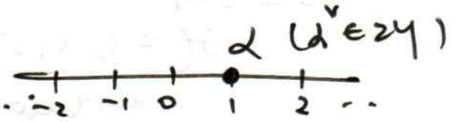
My way of thinking: $\check{\alpha} \in 2Y$ introduces parity issue which 'breaks conjugacy'.

Now is a good time to introduce our main example for today: type A_1. $X = Y = \mathbb{Z}$.

Choice between:

A_1^{sc}. $R = \{2\}$, $R^\vee = \{1\}$.

A_1^{ad} $R = \{1\}$, $R^\vee = \{2\}$



Rmk: when back in p-adic and reductive group settings, usually roots & coroots are flipped (apartment is dual space), but here there is not much difference.

In A_1^{ad}, simple affine reflect^n is $x \mapsto 1-x$, i.e. flip in $\frac{1}{2}$!
Will not be conjugate to 'flip in 0' in W^e due to parity.
(Because $\alpha^\vee$ even, reflect^{n}s will $\pm$ even multiples of α)

More formally: look at possible affine Dynkin diagrams.
If $\alpha^\vee \in 2Y$ then it is long root in type C:

new root contributes another parameter to the affine algebra $L(\tilde{s})$.

$$\ldots = \left((v^{L(s)})^2 - 1 + \theta_{-\alpha}(v^{L(s)+L(\tilde{s})} - v^{L(s)-L(\tilde{s})})\right) \frac{\theta_x - \theta_{s(x)}}{1-\theta_{-2\alpha}} \quad (\check{\alpha} \in 2Y)$$

In type A_1^{sc}:

$$(T_s+1)(T_s - v^2) = 0.$$

$$\theta_1 T_s - T_s \theta_{-1} = (v^2-1)\frac{\theta_1 - \theta_{-1}}{1-\theta_{-2}} = (v^2-1)\theta_1.$$

ome properties of affine Hecke algebra

There is natural W_0-action on $\mathbb{C}[X]$ (from X).

In ideal case, algebra is semi-direct product: "$\theta_x T_s = T_s \theta_{s(x)}$"

Can write relat^{n} in nicer form (based on $\theta_x - \theta_{s(x)}$ on RHS)

$$\theta_x(T_s+1) - (T_s+1)\theta_{s(x)} = \underline{(\theta_x - \theta_{s(x)})G(\alpha)} \in \mathcal{O}$$

(precise significance will be seen later)

$$\text{where } G(\alpha) = \begin{cases} \dfrac{\theta_\alpha (v^{L(s)})^2 - 1}{\theta_\alpha - 1} & (\alpha^\vee \notin 2Y) \\ \cdots & (\alpha^\vee \in 2Y) \end{cases}.$$

E.g. in type A_1, $G(2) = \dfrac{\theta_2 v^2 - 1}{\theta_2 - 1}$. Note $(\theta_1 - \theta_{-1})G(2) = \theta_1 v^2 - \theta_{-1} \in \mathcal{O}$.

~~For~~ Taking $x = 1$ and -1 and adding, see that

$\theta_1 + \theta_{-1}$ commutes with T_s.

Prop. (From Bernstein presentat^{n}),

Center of $\mathcal{H}$ is $\mathcal{O}^{W_0}$, the W_0-invariant subspace of the affine part.

Significance: As usual, study irreps of $\mathcal{H}$ by their central characters, i.e. maximal ideals of $\mathcal{O}^{W_0}$.

From above relation, we see that if we could "divide* by $G(\alpha)$", then

$$\theta_x\left(\frac{T_s+1}{G(\alpha)} - 1\right) = \left(\frac{T_s+1}{G(\alpha)} - 1\right)\theta_{s(x)}$$

which is precisely the semi-direct product structure we want!

Therefore replace $\mathcal{O}$ by its fract^{n} field F to get $\mathcal{H}_F$. (cross-relat^{n}s remain the same)

Prop: $F \rtimes \mathbb{C}W_0 \cong \mathcal{H}_F$ $(= F \otimes_{\mathbb{C}} \mathcal{H}(W_0))$

$$\underset{\substack{\cup \\ t_s}}{\mathbb{C}W_0} \qquad t_s \longmapsto \frac{T_s+1}{G(\alpha)} - 1.$$

*

Go to equivariant K-theory!

From affine to graded Hecke algebras.

Already, many things pointing in same direct^{n}!

① In the first place, X comes from character group.

$$X = \mathrm{Hom}(T, \mathbb{C}^{\times}).$$

Affine part $\mathbb{C}[X]$ can be identified with

$\mathcal{O}(T)$ = polynomial f^{n}s / regular f^{n}s on torus T "Lie gp!"
(coordinate ring)

② Central character $\longleftrightarrow$ max. ideal $\longleftrightarrow$ pts in gp T.

Here, central character $\longleftrightarrow$ W_0-orbits in $T \times \mathbb{C}^{\times}$

param of v.

③ Quotient field $F \cong$ rational f^{n}s on torus T.

④ Params in exponents not ideal. Want to "differentiate" so that params appear as constants.

Considerat^{n}s:

① Lie gp $\rightsquigarrow$ Lie alg: 'localise' at a point of T.
$\longleftrightarrow$ maximal ideal I.

② However, must carry over W_0-action, i.e. I is ideally W_0-inv.
i.e. localise at W_0-invariant point.

③ $\mathbb{C}^{\times}$ part is controlled by $v \to$ params.
"Hard part": finite Hecke alg. Great if we can recover $\mathbb{C}W_0$
i.e. $v \mapsto 1$, or $v - 1 \in I$.

④ Conclusion: localise at $(\tau_0, 1) \in T \times \mathbb{C}^{\times}$, $W_0 \cdot \tau_0 = \tau_0$.

How to localise?

Recall in alg geom: tangent space I/I^2.

In fact natural construct^{n} in alg geom:
(sometimes see it in context called power series expans^{n} at a pt)

graded alg: $\mathcal{O}/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$
is a polynomial ring in $\dim(T \times \mathbb{C}^{\times})$ variables! (True whenever we localise at a smooth pt)

Have passed to "Lie alg": this polynomial ring is ~~the~~ identified w/ polynomial/regular f^{n}s on $\mathfrak{t} \oplus \mathbb{C}$ (vector space).

Def^{n}: Graded Hecke algebra $\mathbb{H} = \bigoplus I^{i}\mathcal{H} / I^{i+1}\mathcal{H}$.

Def^{n} of graded Hecke algebras.

Rmk: T_1 t are on the y-side

Def^{n}/Prop: (Graded Hecke algebra).

"affine part" ↓ "finite part" ↓

$$\mathbb{H} = \underbrace{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}_{\cong\ \mathcal{O}(\mathfrak{t})\otimes_{\mathbb{C}}\mathbb{C}[r]} \otimes_{\mathbb{C}} \mathbb{C}[W_0]$$

Notation: write $\overline{\mathcal{O}}$ for affine part. ←

$\mathcal{O}(\mathfrak{t})\otimes_{\mathbb{C}}\mathbb{C}[r]$ ← grading comes fr. here (cf. homology, as we will see later)

↑ indeterminate $r = v-1 \mod I^2$

Again, remains to specify the cross-relations.
we do it in A_1: (assume localise at $(1,1)\in T\times\mathbb{C}^\times$)
(note 1 is always W_0-inv.)

Recall In affine relat^{n}:

$$(\theta_1-1)T_s - T_s(\theta_{-1}-1) = ((\theta_1-1)-(\theta_{-1}-1))(G(\alpha)-1)$$
$$= ((\theta_1-1)-(\theta_{-1}-1))\frac{v^2-1}{-(\theta_{-2}-1)}$$

Rmk: $(v^{L(s)})^2$

"Reduce mod I^2:"

$$\overline{\theta_1-1}\,t_s - t_s(\overline{\theta_{-1}-1}) = (\overline{\theta_1-1} - \overline{\theta_{-1}-1})\frac{2\overline{v-1}}{\overline{\theta_2-1}}$$

Rmk: $2L(s)\overline{v-1}$

Indeed the relat^{n} is:

$$\phi(t_s+1) - (t_s+1)s(\phi) = (\phi - s(\phi))\,g(\alpha)$$

$$g(\alpha) = \begin{cases} \dfrac{2rL(s)}{\overline{\theta_\alpha-1}}+1 & (\alpha^\vee \notin 2Y) \\ \ldots & (\alpha^\vee \in 2Y) \end{cases}$$

Rmk: In graded, no longer have to deal with extra parameter from $\alpha^\vee\in 2Y$ b/c params just consts. Just need one parameter per simple root in R.

Rmk: In general def^{n} of graded, param here is $c = 2L(s)$ (note analogies to affine!) → For each simple α, a param $c(\alpha)$ & replace $2L(s)$ w/ $c(\alpha)$.

Some properties of graded Hecke algebras:

Prop: Center of $\mathbb{H}$ is $\overline{\mathcal{O}}^{W_0}$.
Replace $\overline{\mathcal{O}}$ w/ fract^{n} field $\overline{F}$:

Prop: $\overline{F}\rtimes\mathbb{C}W_0 \cong \mathbb{H}_{\overline{F}}$ $(=\overline{F}\otimes_{\mathbb{C}}\mathbb{C}W_0)$

$$t_s \longmapsto \frac{t_s+1}{g(\alpha)} - 1.$$

Def^{n}: (general graded Hecke algebra)

$$\phi t_s - t_s s(\phi) = (\phi - s(\phi))\frac{c(\alpha)r}{\alpha}$$

(c respects W-conjugat^{n})

*

Go to equivariant homology!

Reducing ~~from~~ affine to graded: (Lusztig's second reduct$^{\underline{n}}$ thm)

→ will see why second first later

Semi-direct product hints at a strong relationship between $\mathcal{H}$ and $\mathbb{H}$!

$\exp : \mathfrak{t} \to T$.

rat$^{\underline{al}}$ f$^{\underline{n}}$s on torus T ←

↑ How to relate Lie gp & Lie alg? Exponential map!

$$\begin{tikzcd}
F \rtimes \mathbb{C}W_0 \arrow[r, hook, "\sim"] \arrow[d, "\exp^*"'] \arrow[d, shift left=3, "\mathrm{id}"] & \mathcal{H}_F \supset \mathcal{H} \\
\bar{F} \rtimes \mathbb{C}W_0 \arrow[r, hook, "\sim"] \arrow[d, "\exp^*"] & \mathbb{H}_{\bar{F}} \supset \mathbb{H} \\
\hat{\bar{F}} \rtimes \mathbb{C}W_0 \arrow[r, hook, "\sim"] & \hat{\mathbb{H}}_{\hat{\bar{F}}} \supset \hat{\mathbb{H}} \\
 & \hat{\bar{O}} \otimes_{\mathbb{C}} \mathbb{C}W_0 \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

→ Rmk. also Lusztig mentions Chern character from K-theory to homology which involves exponential map.

↓ rat$^{\underline{al}}$ f$^{\underline{n}}$s on v.s $\mathfrak{t}$. ←

Of course, exp is not rational!

Therefore, must work in complet$^{\underline{n}}$ to holomorphic/meromorphic f$^{\underline{n}}$s (in Lusztig, formal power series)

↳ again, in alg. geom., taking complet$^{\underline{n}}$ at maximal ideal = power series ring

↖ meromorphic f$^{\underline{n}}$s on $\mathfrak{t}$.

↖ holomorphic f$^{\underline{n}}$s on $\mathfrak{t} \oplus \mathbb{C}$.

Of course, this does not give us anything yet, it is still too general.

However, recall we look at irreps by central characters, so we just need to get correspondence between central characters = maximal ideals, which kills a lot of the noisy extra informat$^{\underline{n}}$ in the algebras above.
modding out by

$$\text{Central characters in } \mathcal{H} \xleftrightarrow{\sim} \begin{array}{c} W_0\text{-orbits} \\ (\Sigma, v_0) \in T \times \mathbb{C}^\times \end{array} \xleftrightarrow{\sim} \begin{array}{c} \text{Max ideals} \\ (\text{of } O^{W_0}) \\ J_{\Sigma, v_0}. \end{array}$$

$\uparrow \exp$.

$$\text{Central characters in } \mathbb{H} \xleftrightarrow{\sim} \begin{array}{c} W_0\text{-orbits} \\ (\bar{\Sigma}, r_0) \in \mathfrak{t} \oplus \mathbb{C} \end{array} \xleftrightarrow{\sim} \begin{array}{c} \text{Max ideals} \\ (\text{of } \bar{O}^{W_0}) \\ \bar{J}_{\bar{\Sigma}, r_0}. \end{array}$$

Note modding out by J results in a f.d. algebra (the F part becomes just 'the values of f$^{\underline{n}}$s at points in the W_0-orbit')

Main Thm: Under condit$^{\underline{n}}$s on $\bar{\Sigma}, r_0$, Above maps induce an isomorphism

$$\mathcal{H} / J_{\Sigma, v_0}\mathcal{H} \xrightarrow{\sim} \hat{\mathbb{H}} / \hat{\bar{J}}_{\bar{\Sigma}, r_0}\hat{\mathbb{H}} = \mathbb{H} / \bar{J}_{\bar{\Sigma}, r_0}\mathbb{H}$$

(and hence a biject$^{\underline{n}}$ between irreps of corresponding central character)

Pf of Main thm.

$$\begin{tikzcd}
F \rtimes \mathbb{C}W_0{}_{t_s} \arrow[r, hook, "\sim"] \arrow[d] & \mathcal{H}_F \arrow[r, phantom, "\supset"] & \mathcal{H}_{\frac{T_s+1}{G(\alpha)}-1} . \\
\hat{F} \rtimes \mathbb{C}W_0{}_{t_s} . \arrow[r, hook, "\sim"] & \hat{\mathbb{H}}_{\hat{F}} \arrow[r, phantom, "\supset"] & \hat{\mathbb{H}}_{\frac{t_s+1}{g(\alpha)}-1} .
\end{tikzcd}$$

In other words, the induced homomorphism sends

$$\underset{\mathcal{H}}{\underset{\cap}{T_s+1}} \longmapsto \underset{\hat{\mathbb{H}}_{\hat{F}}}{\underset{\cap}{(t_s+1)}}\frac{G(\alpha)}{g(\alpha)} .$$

Therefore suffice to show that $\frac{G(\alpha)}{g(\alpha)}$, as a meromorphic f^n on $\mathfrak{t}\oplus\mathbb{C}$, is a) defined (i.e. holomorphic), so that it lies in $\hat{\mathbb{H}}/\hat{\mathcal{J}}\hat{\mathbb{H}}$ in the first place

b) non-zero, so that ~~it is invertible in~~ the map to the f.d. $\hat{\mathbb{H}}/\hat{\mathcal{J}}\hat{\mathbb{H}}$ is invertible (recall the basis elements are essentially f^n values on $(\bar{\Sigma}, r_0)$).

on the W_0-orbit $(\bar{\Sigma}, r_0)$.

This is just a computat^n. We do it in our A_1 example:

$$G(\alpha) = \frac{\theta_2 v^2 - 1}{\theta_2 - 1}, \quad g(\alpha) = \frac{2r + \theta_2 - 1}{\theta_2 - 1} .$$

Recall $G(\alpha)$ is a f^n on the torus but we pull it back to $\mathfrak{t}\oplus\mathbb{C}$ via the exponential map.

As a f^n on $\mathfrak{t}\oplus\mathbb{C}$ with variables (x, r),

$$\frac{G(\alpha)}{g(\alpha)} = \frac{e^{2x+2r}-1}{e^{2x}-1} \cdot \frac{2x}{2r+2x} = \frac{e^{2x+2r}-1}{2x+2r} \Big/ \frac{e^{2x}-1}{2x}$$

a) Defined: just need $2x \neq 2\pi i k$ $(k \neq 0)$.

b) Non-zero: just need $2x + 2r \neq 2\pi i k$ $(k \neq 0)$.

$\forall (x, r) \in (\bar{\Sigma}, r_0)$.

Clearly then, so long as x, r are real, we are done.

This is the condit^n we need in the main thm.! $\square$.

(known as real infinitesimal character).

Cor. There is a biject$^{\underline{n}}$ ~~between~~

$$\mathrm{Irr}_{(W_0\cdot t,\, e^r)}\, \mathcal{H} \xrightarrow{\sim} \mathrm{Irr}_{(W_0\cdot \log t,\, r)}\, \mathbb{H}$$

↳ tells you what v acts as, which is usually real > 0.

for t in the $\mathbb{R}_{>0}$ part of the torus T and r real.

General reduct$^{\underline{n}}$ to graded Hecke.

In general, how to reduce? Two steps.

① Need to choose a W_0-inv. point T_0 to localise at (so far, $T_0 = 1$). Considering 1 is always W_0-inv. & the act$^{\underline{n}}$ on T often looks like $z \mapsto 1/z$, we try to find T_0 in the unitary part of the torus.

~~②~~ i.e. ~~we write~~ Given a W_0-orbit $W_0 t \subset T$, write

$$t = \underset{\text{unitary}}{t_c} \cdot \underset{\mathbb{R}_{>0}}{t_h} \quad \text{(polar decomposit}^{\underline{n}}\text{ in torus)}.$$

So long as t_c is now W_0-invariant we can do exactly ~~③~~ the same thing for Lusztig's second reduct$^{\underline{n}}$ thm. (Just replace exp with $t_c \cdot \exp$)

② How to ensure t_c is W_0-invariant?

Lusztig's first reduct$^{\underline{n}}$ thm.

We reduce the root system R to a smaller one $\dot{R}$ with corresponding $\dot{W}_0$ s.t. t_c is $\dot{W}_0$ invariant (by def$^{\underline{n}}$) and corresponding affine Hecke $\mathcal{H}'$.

Thm. There is a biject$^{\underline{n}}$

$$\mathrm{Irr}_{(W_0\cdot t,\, v_0)}\, \mathcal{H} \xrightarrow{\sim} \mathrm{Irr}_{(\dot{W}_0\cdot t,\, v_0)}\, \mathcal{H}'.$$

Pf ~~is~~ involves technicalities on root systems (and not so much on graded Hecke).

Essentially it still goes by an alg. isomorphism which induces the biject$^{\underline{n}}$ of irreps. □

Equivariant K-theory approach & why it fails.

(Ref: Chriss & Ginzburg)

G cpx s.s. gp.

$\mathcal{H} \cong K^{G\times\mathbb{C}^\times}(Z)$

Naively, only depends on G: no extra parameters!

Very rough outline: $\curvearrowleft$ Steinberg variety.

Key idea to relate the geometry:

$$K^{G\times\mathbb{C}^\times}(T^*\mathcal{B}) \underset{\uparrow}{\cong} K^{B\times\mathbb{C}^\times}(pt) \cong R(T\times\mathbb{C}^\times) \cong R(T)[~~q, q^{-1}~~ v, v^{-1}]$$

K-theoretic $\uparrow$ affine part of $\mathcal{H}$!

i.e. v represents trivial rep. of $\mathbb{C}^\times$ geometrically. (coordinate ring of T after $\otimes\mathbb{C}$)

W_0-action ~~on~~ comes K-theoretically from $K^{G\times\mathbb{C}^\times}(Z)$ on $K^{G\times\mathbb{C}^\times}(T^*\mathcal{B})$.

Then $\mathcal{H}$ and $K^{G\times\mathbb{C}^\times}(Z)$ have same act^n on $K^{G\times\mathbb{C}^\times}(T^*\mathcal{B})$ (convolution action).

$R(G\times\mathbb{C}^\times) \cong \mathcal{O}^{W_0}$ (center of $\mathcal{H}$) acts naturally on $K^{G\times\mathbb{C}^\times}(Z)$

Also have K-theoretic X (affine) & W_0-act^{n}s on $K^{G\times\mathbb{C}^\times}(Z)$

Get a $\mathcal{H}$ act^n on $K^{G\times\mathbb{C}^\times}(Z) \cong$ reg rep. of $\mathcal{H}$.

gross over-simplificat^n

Here the indeterminate v represents the trivial rep. of $\mathbb{C}^\times$ geometrically: parameter v which corresponds to $\mathbb{C}^\times$-act^n on Z is not really related to the various reflect^{n}s in W_0. —

$\rightarrow$ Go to "From affine to graded"

What makes equivariant homology amenable nice?

~~Idea:~~ Two key points:

① Finite/W_0-part is just $\mathbb{C}W_0$, so to specify W_0-act^n just need group act^n.

② Parameters come as constants only!

Very rough outline:

Same as ~~=~~ K-theory: want a $\mathbb{H}$-action on $H_*^{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}}_N, \ddot{\mathcal{L}})$

G cpx gp, in addit^n have nilp. orbit C (in Levi)

'Steinberg variety' $\swarrow$

$\rightarrow$ local sys on $\dot{\mathfrak{g}}_N$.

- W_0-action comes from intersect^n cohomology: $w \to \mathrm{Aut}\, H_*^{G\times\mathbb{C}^\times}$
- ~~Affine~~ action comes from homological cup product:

$\downarrow$ $\mathcal{O}(t\oplus\mathbb{C}) = \bar{\mathcal{O}} \cong H^*_{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}})$ $\curvearrowleft$ 'flag variety'

The parameters, which are just constants, in fact come from this part of the act^n: from the nilpotent orbit we obtain ~~geometric parameters~~ parameters which are reflected in a character $\psi: M_L(\phi_0) \longrightarrow \mathbb{C}^\times$ $(M)_0$ $\cong T\times\mathbb{C}^\times$ sth like centralizer of nilp. ele. (acting on ^{n}ker(ad x_0^{c-2})) for each simple root α

Go to next pg.

after a series of reduct^{n}s, reduce to the case of a maximal parabolic corr. to α.

Recall e.g. ~~and we~~ in A_1, relat^n is $\alpha t_s + t_s\alpha = 2(2)r$.

$(\ker\chi = M_1)$

~~$\ker$~~ $d\chi =$ ~~M_1~~ $\alpha - cr$

$\ker d\chi = m_1$

and we reduce the homology to

$$H^*_{M_1^\circ}(pt) \cong \mathcal{O}(m_1) \cong \bar{\mathcal{O}}/(\alpha - cr)$$

Parameters obtained are called geometric parameters

Also, affine part is simpler (just polynomial ring)!

gross over-simplificat^n

*Because params are attached to α, and $\alpha \in X$ belongs to the affine side (is viewed as a function on the affine side) i.e. can control the params on the affine side.

$\rightarrow$ Go to "Reducing affine to graded"

For each simple root α, recall in A_1 case relat^n is $\frac{\alpha}{2}t_s + t_s\frac{\alpha}{2} = 2r = cr$.

Reduce to the case of maximal parabolic corresponding to α.

From the nilp. orbit, the param is reflected in a character

$\chi : M \to \mathbb{C}^\times$ (acting on $\mathfrak{n}/\ker(\mathrm{ad}\,x_0^{c-2})$).

$M \cong T\times\mathbb{C}^\times$ 'centraliser of nilp. ele in nilp. orbit'

$d\chi = \alpha - cr$.

$M_1 = \ker\chi$

After a series of reduct^{n}s,

$H^*_{\underset{M_1}{\cancel{G\times\mathbb{C}^\times}}}(pt) \cong \mathcal{O}(\mathfrak{m}_1) \cong \bar{\mathcal{O}}/\langle\alpha - cr\rangle$.

So in the reduced version α acts as cr, and this is where the param comes from. Params obtained in this way are called geometric params.

Key pt: control param on the affine side!

→ params are attached to α, and $\alpha\in X$ belongs on the affine side

→ Affine side is simpler! (just a polynomial ring).

Equiv. K-theory. G cpx gp.

$$\begin{tikzcd} \mathcal{O}^{W_0} \subset Z\mathcal{H} \arrow[r, phantom, "\circlearrowright"] & K^{G\times\mathbb{C}^\times}(Z) \\ R(G\times\mathbb{C}^\times) \arrow[u, phantom, "\cong"] \arrow[ur, bend right] & \end{tikzcd}$$

- have W_0-act^n and X-act^n (act^n of the affine part).

v corresponds to trivial rep. of $\mathbb{C}^\times$.

Not really related to reflect^{n}s in W_0: params are tied up in the finite Hecke part.

Equiv. homology. G cpx gp. + nilp. orbit $\mathcal{C}$ on Lie.

$\mathcal{H} \circlearrowright H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{L})$. - have W_0-act^n (just gp. act^n from intersect^n cohomology) and X-act^n fr. cup. product ($\bar{\mathcal{O}} \cong H^*_{G\times\mathbb{C}^\times}(\mathfrak{g})$)

Now consts are just params, so can control it from affine side.

E.g. in A_1, $\frac{\alpha}{2}t_s - t_s(-\frac{\alpha}{2}) = 2r = cr$.

So suff. show α acts as cr ~~in~~ after reducing the homology (and s as id) to a simpler version.

Concrete example

$\mathcal{H}$ = Iwahori-Hecke algebra of $SL_2(\mathbb{Q}_p)$ $(C_c^\infty(I\backslash G/I))$

↳ is of type A_1^{ad} (in our notat^n) two parameters of value 1 each

↳ Recall relat^n is of form $(T_s+1)(T_s-p)=0$. So will specialise $v \mapsto \sqrt{p}$.

$\mathbb{H}$ = associated graded Hecke algebra after localising at $(1,1)$.

↳ Corresponding root system is clearly A_1

↳ We did not mention this (as this is the ~~type~~ $\check{\alpha} \in 2Y$ case) but the parameter is value 2 (trust me!)

(1+1) ↳ turns out this is a geometric param!

Irreps of $\mathcal{H}$ w central char $(\{t, 1/t\}, \sqrt{p})$ $t \in \mathbb{R}_{>0}$ $\xleftrightarrow{\sim}$ Irreps of $\mathbb{H}$ w central char $(\{\pm \log t\}, \frac{1}{2}\log p)$ $\xrightarrow{\sim}$ $G = SL_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2$ (& assoc. nilp-orbit triples $(\xi, y, V) = \{0\}$)

$\{\pm \log t\}$ $=$ W_0-orbit in $\mathfrak{t}$ $\overline{\Sigma}$

$\frac{1}{2}\log p$ $=$ r

ξ s.s. in $\mathfrak{g}$, y nilp. in $\mathfrak{g}$ } up to G-conj.

(V not impt: an irrep of component gp. of a centraliser satisfying some properties; in our case, only 1 possibility (sgn of $\mathbb{Z}_2$))

s.t. $[\xi, y] = 2ry$.

For $t \in \mathbb{R}_{>0}$, there is one irrep if $t \neq p, 1/p$: in fact it is $\operatorname{ind}_{\mathcal{O}}^{\mathcal{H}} \mathbb{C}_t$ ~~where~~ $\cong \operatorname{ind}_{\mathcal{O}}^{\mathcal{H}} \mathbb{C}_{t^{-1}}$ (dim 2).

$\mathcal{O} = \mathbb{C}[X]$

where θ_1 ~~t~~ acts as ~~p~~ t.

↳ principal series modules (irred is ~~comp. factor~~ subquotient)

Then if $t = p, 1/p$, there are two irreps! & they are the trivial & Steinberg rep. of SL_2!

In fact we have s.e.s.

$0 \to St \to \operatorname{ind}_{\mathcal{O}}^{\mathcal{H}} \mathbb{C}_p \to triv \to 0$

Why? Tracing through, we see that $\theta_1 = p^{-1} T_{s'_\alpha} T_{s_\alpha}$ (I-M presentat^n)

In triv, $T_{s'_\alpha}$, T_{s_α} act as p.

In St, $T_{s'_\alpha}$, T_{s_α} act as -1.

$\xleftrightarrow{\sim}$

Recall now s.s. orbits in $\mathfrak{sl}_2$ $\xleftrightarrow{\sim} \mathfrak{h}/W \xleftrightarrow{\sim} \mathbb{C}/\{\pm 1\}$: reminds you of W_0-orbits in $\mathfrak{t}$

① $y = 0$: $\xi = \begin{pmatrix} a & \\ & -a \end{pmatrix}$ $(a \in \mathbb{R}_{\geq 0})$ controls the W_0-orbit $\overline{\Sigma} = \{a, -a\}$.

② $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, ~~$\mathcal{F}$~~ $\xi = \begin{pmatrix} r & \\ & -r \end{pmatrix}$ infact gives another W_0-orbit $\overline{\Sigma} = \{\pm \log p\}$!

(Rmk: off by factor of 2 due to the fact that when defining graded, take $\alpha = 2$, but here, $\alpha = 1$ $(\check{\alpha} \in 2Y)$!)

What happens is the Iwahori-spherical here is actually a subalgebra of the easier (1-parameter one from A_1^{sc}) via $\theta_1 \mapsto \theta_1^2$.

What about negative characters? ($t < 0$)

We localise at $(-1, \cancel{1})$. But now be careful! If you trace through, you see that the parameter is now 0! i.e. $\mathbb{H}$ ~~is~~ is nothing but $\overline{\mathcal{O}} \rtimes W_0$.

So the behaviour changes! When $t = 1$ $(1-1)$ there are two irreps (coming from W_0).

In fact $\operatorname{ind}_{\mathcal{O}}^{\mathcal{H}} \mathbb{C}_{-1} \cong (-triv) \oplus (-St)$ (central char $\{-1\}$). For other t, there is only one irrep.

↑ ↑ same as triv (resp. St) when restricted to $\mathcal{H}(W_0)$.

∃ other names e.g. + −