Bernstein-Gelfand-Gelfand category $\mathscr{O}$: key ideas and results...

... or, a sales pitch for $\mathscr{O}$

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Disclaimer

- Category $\mathcal{O}$ is algebraic, technical;
- 20 mins is not enough to describe everything in detail:
  - We want to get to the elegant results about $\mathcal{O}$ and understand why $\mathcal{O}$ is so interesting!
- Focus on the coloured boxes
  1. Look out for

     Theorem

     they are our highlights!
  2. We'll mostly skip

     Proof sketch

     except to highlight certain interesting things.
  3. Where there are good references, additional details can be found in

     Reference

Why category $\mathscr{O}$?

- Natural extension of the main focus of MA5211: structure theory of complex semisimple Lie algebras and their representations
- Deep connections to many areas of Lie theory and representation theory
- Elegant results and interesting, difficult questions!

- Some say that when BGG discovered it and realised how nice it is, they exclaimed, "$\mathscr{O}$!"
  - $\mathscr{O}$ stands for *basic* in Russian

Preliminaries

Recall (Setup)

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$:
complex semisimple Lie algebra and triangular decomposition

$Z(\mathfrak{g}) \subset U(\mathfrak{g})$:
center of universal enveloping algebra

Δ	$\subset \Phi(= R)$	$\subset \Lambda(= P)$	$\subset \mathfrak{h}^*$
simple roots	$\subset$ root system	$\subset$ (integral) weight lattice	$\subset$ weights

$\rho =$ half-sum of positive roots $=$ sum of fundamental weights

W: Weyl group

Motivations

Recall

Nice properties of finite-dimensional $U(\mathfrak{g})$-mods:

- Weight space decomposition
- Highest weight vector

Above properties depend on finite-dimensionality of modules.

However, for each weight λ we have a **Verma** module $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ of fundamental importance (universal highest weight module), which is *infinite-dimensional*.

So, now, want to consider infinite-dimensional modules.

(Also, f.d. modules are all completely reducible, so "not very interesting".)

Preliminaries

Definition (Category $\mathcal{O}$)

$\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-mod with objects M satisfying:

- ($\mathcal{O}1$) M is finitely generated as $U(\mathfrak{g})$-mod
  (natural finiteness condition)
- ($\mathcal{O}2$) M has weight space decomposition
- ($\mathcal{O}3$) M is locally $\mathfrak{n}$-finite, i.e. $U(\mathfrak{n})v$ is f.d. for all $v \in M$
  (ensures existence of highest weight vector)

Preliminaries

Of course, as with most other definitions via axioms, category $\mathcal{O}$ is useful because it satisfies many more nice properties.

Proposition (Properties of Category $\mathcal{O}$)

1. *$\mathcal{O}$ is closed under submodules, quotients, finite direct sums, and is an abelian category.*
2. *$\mathcal{O}$ is Noetherian.*
3. *Each $M \in \mathcal{O}$ has a finite filtration with factors being highest weight modules.*
4. *$\mathcal{O}$ is Artinian.*

In other words $\mathcal{O}$ satisfies all the nice properties (think Jordan-Holder, Krull-Schmidt) that we could ever hope for.

Proof sketch

2. $U(\mathfrak{g})$ is Noetherian (its associated graded ring is commutative f.g., hence Noetherian). (We need this for (1) on submodules as well)
3. Successively quotient out by the highest weight module generated by a highest weight vector in a fixed generating set. The finiteness conditions ensures this terminates.
4. By (3), suffice to show the Verma $M(\lambda)$ is Artinian. This can be proved once we know more about central characters.

Proposition (Simple objects in $\mathscr{O}$)

The simple objects in $\mathscr{O}$ are the $L(\lambda)$ (unique simple quotient of the Verma $M(\lambda)$). They are f.d. iff λ is dominant integral.

1. Blocks
   - Reduce $\mathcal{O}$ to smaller, more manageable subcategories

As usual, we would like to parameterise representations by their central character.

Highest weight modules all have central character. We may thus associate to each weight λ a character χ_λ.

We will need one preliminary, standard result.

Theorem (Harish-Chandra)

Central characters are parameterized by dot-linkage classes of weights, which is a ρ-shifted action of W on weights:

- $\chi_\lambda = \chi_\mu$ iff λ, μ are dot-linked, i.e., $\lambda = w \cdot \mu := w(\mu + \rho) - \rho$
- Every central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form χ_λ.

Reference

Humphreys chapter 23. It is closely related to the Chevalley restriction theorem.

Blocks

Since each weight space is f.d., we then have a generalised eigenspace decomposition by central characters:

Proposition (Decomposition by central characters)

$M = \oplus_\chi M_\chi$, where
$M_\chi = \{v \in M | (z - \chi(z))^n v = 0$ for $z \in Z(\mathfrak{g})$ and n depending on $z\}$

In other words $\mathscr{O}$ decomposes as a direct sum of subcategories

$$\mathscr{O} = \oplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathscr{O}_{\chi_\lambda}.$$

We write $\mathscr{O}_{\chi_\lambda} =: \mathscr{O}_\lambda$.

Remark

The fact that each $M \in \mathscr{O}$ has a filtration by highest weight modules gives another motivation for looking at such decomposition.

Blocks

Of course, this is not necessarily the block decomposition of $\mathscr{O}$.

In particular, $\mathscr{O}_\lambda$ may be further decomposable.

However, in the case of *integral* weights, it turns out that the $\mathscr{O}_\lambda$ are indeed indecomposable.

Proposition (Block of $\mathscr{O}$ for integral weights)

For λ integral, $\mathscr{O}_\lambda$ is a block of $\mathscr{O}$.

The proof relies on a preliminary lemma which is in fact essential to the development of $\mathcal{O}$.

Proposition (Embedding of Vermas, s simple)

Suppose λ integral, s simple and $s \cdot \lambda < \lambda$. Then there is a proper embedding $M(s \cdot \lambda) \subset M(\lambda)$.

Proof sketch

- If v^+ is highest weight vector in $M(\lambda)$, then $y_\alpha^n v^+$ is highest weight (of the correct weight $s \cdot \lambda$)
  - Follows directly from standard relations completely within $U(\mathfrak{g})$.
- This gives a map $M(s \cdot \lambda) \to M(\lambda)$.
  - But in fact all nonzero maps between Vermas must be embeddings, because they are free of rank 1 as $U(\mathfrak{n}^-)$-modules and $U(\mathfrak{n}^-)$ has no zero divisors.

Reference

Humphreys 21.2 for the relations in $\mathfrak{g}$ mentioned above.

Proposition (Embedding of Vermas, s simple)

Suppose λ integral, s simple and $s \cdot \lambda < \lambda$. Then there is a proper embedding $M(s \cdot \lambda) \subset M(\lambda)$.

Proposition (Block of $\mathscr{O}$ for integral weights, cont.)

For λ integral, $\mathscr{O}_\lambda$ is a block of $\mathscr{O}$.

Proof sketch

From the embedding: $L(s \cdot \lambda)$, $L(\lambda)$ occur as subquotients of the indecomposable(highest weight) $M(\lambda)$.
Iterating over simple s, $L(\lambda)$, $L(w \cdot \lambda)$ lie in the same block for all w.

Remark (On non-integral weights)

In the non-integral case, $\mathscr{O}_\lambda$ is in general further decomposable.
The issue is that $\mathfrak{g}$ only acts by integral multiples of roots, hence we can decompose any module by cosets modulo the root lattice.

While the conceptual idea is the same, we do not want to deal with the added technicalities involved, so **from now on we will only ensure that things work in the integral case**.

(For f.d. modules, all weights are integral, so it is also not uncommon to just call them the *weights*.)

Remark (Grothendieck group of blocks)

With notion of block $\mathscr{O}_\lambda$:
$[L(w \cdot \lambda)]$ ($w \in W$) forms a $\mathbb{Z}$-basis of the Grothendieck group $K(\mathscr{O}_\lambda)$.
So too do the $[M(w \cdot \lambda)]$ (since the multiplicity of $L(w \cdot \lambda)$ is 1 and all other factors have weight $< w \cdot \lambda$)

Digression: Weyl character formula

BGG approach: Weyl character formula is essentially a statement about change-of-basis between $[M(w \cdot \lambda)]$ and $[L(w \cdot \lambda)]$.

Theorem (Weyl character formula, BGG formulation)

For λ dominant integral, $[L(\lambda)] = \sum_{w \in W} (-1)^{l(w)} [M(w \cdot \lambda)]$
or in other words, $\quad \mathrm{ch}\ L(\lambda) = \sum_{w \in W} (-1)^{l(w)} \mathrm{ch}\ M(w \cdot \lambda)$

Remark

We will see later that the WCF in fact has even a higher (homological) interpretation in the BGG setting: the BGG resolution.

2 Translation functors

- How are blocks related?

Blocks

We would now like to investigate the relation between the blocks of $\mathscr{O}$.

To do so we need functors between the $\mathscr{O}_\lambda$, i.e. shifting the weights of the modules.

The natural way is by taking **tensor products**.

Tensor products

First, some terminology:

Definition (Standard filtration)

$M \in \mathcal{O}$ is said to have a **standard filtration** (SF) if it has a filtration with successive quotients being Vermas.

Remark

This notion is natural because:

Recall

- Every module in $\mathcal{O}$ already has filtration by highest weight modules
- Vermas form $\mathbb{Z}$-basis for $K(\mathcal{O})$

Tensor products

We want to look at what happens to Vermas under tensor product.

Proposition (Standard filtration of Verma $\otimes$ f.d.)

If L is f.d., then $M(\lambda) \otimes L$ has a standard filtration with factors of the form $M(\lambda+\mu)$, where μ are the weights of L (with multiplicity).

Proof sketch

- Key point: $M(\lambda) \otimes L$ is an induced module
  - $M(\lambda) \otimes L = (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda) \otimes L \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes L)$
- Any induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ has standard filtration:
  - Order the weight vectors of N in increasing order of weight to obtain a filtration of $U(\mathfrak{b})$-modules;
  - Tensoring up to $U(\mathfrak{g})$ gives the desired standard filtration

Remark

This result will also be important when we talk about projectives.

Translation functors

How do we bring weights in $\mathcal{O}_\lambda$ to weights in $\mathcal{O}_\mu$ by tensoring with a f.d. module?

The natural choice is $V := L(w(\mu - \lambda))$ with $w(\mu - \lambda)$ dominant (to ensure f.d.).

Definition (Translation functors)

Suppose λ, μ integral. The translation functor $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is defined by

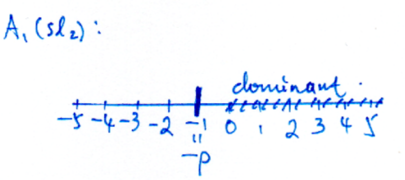
$$T_\lambda^\mu = \pi_\mu \circ (V \otimes (-)) \circ \iota_\lambda$$

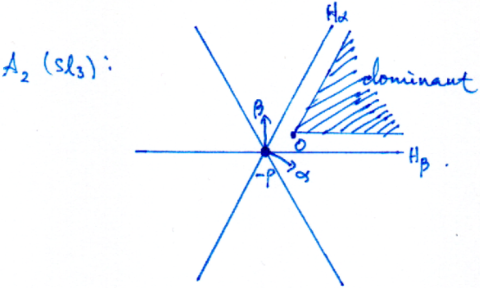
where π_μ, ι_λ are the obvious inclusions and projections to and from $\mathcal{O}$.

Translation functors

Whenever λ is *dominant* integral, its dot-linkage class is of size $|W|$, and $K(\mathscr{O}_\lambda)$ is of rank $|W|$.

We expect all such $\mathscr{O}_\lambda$ to 'look the same'.





Translation functors

Theorem (Equivalence of blocks)

For any dominant integral λ, T_0^λ and T_λ^0 are left/right adjoint and they induce an equivalence of categories between $\mathcal{O}_0$ and $\mathcal{O}_\lambda$.

Proof sketch

- Left/right adjoint:
  - Follows from the fact that the corresponding V's are duals of each other.
- Key point: $T_\lambda^0 M(w \cdot \lambda) \cong M(w \cdot 0)$ and $T_0^\lambda M(w \cdot 0) \cong M(w \cdot \lambda)$
  - We know the exact weights occurring in the standard filtration after tensoring the Verma
- Equivalence:
  - Now follows from the fact that T_0^λ, T_λ^0 induce isomorphisms of the Grothendieck groups and abstract nonsense.

Principal block

Consequently, it is of no harm+most interest to restrict attention to:

Definition (Principal block)

$\mathcal{O}_0$ is called the **principal block**. (It contains the trivial module.)
$\mathcal{O}_0$ has only $|W|$ many simples $L_w := L(w \cdot 0)$ and corresponding Vermas $M_w := M(w \cdot 0)$ (and Grothendieck group of finite rank $|W|$).

This is nice because it means that we can study $\mathcal{O}_0$ in the spirit of finite-dimensional algebras.

Remark (Singular weights)

For other non-dominant weights (weights that lie on hyperplanes): in some sense, they are less complex than $\mathcal{O}_0$ (e.g. they have $< |W|$ simples). Once $\mathcal{O}_0$ is known, similar application of translation functors allows us to determine the same for the other weights.

3 Projectives

Cartan matrix

Recall

In the classical case (f.d. algebras),

- {simple modules}$\overset{\sim}{\longleftrightarrow}${projective indecomposable modules (PIMs)}
- **Cartan matrix** C: key info about the composition factor multiplicities of the PIMs.
  (This is different from the Cartan matrix in root systems - Cartan is too famous!)

We will see that, much as in the modular representation theory for finite groups, C for $\mathscr{O}_0$ can be written in the extremely nice form $C = D^T D$, with Vermas playing the central role; this is the celebrated **BGG reciprocity**.

Projectives in $\mathcal{O}$

Proposition (Projectives in $\mathcal{O}$)

$\mathcal{O}$ has enough projectives.

Proof sketch

Of course, first we need to find some projectives in $\mathcal{O}$. In fact:

- If μ is maximal in its dot-linkage class, then $M(\mu)$ is projective
  - Given a map $M(\mu) \to B$ and a surjection $A \to B$, by maximality of μ the preimage of B's HWV in A must be a HWV which allows us to pullback the map to $M(\mu) \to A$
- $M(\mu) \otimes L$ is still projective for f.d. L
  - Use the tensor-hom adjunction and the characterisation of projectives by exactness of Hom.

Projectives

Proposition (Projectives in $\mathscr{O}$)

$\mathscr{O}$ has enough projectives.

Proof sketch (cont.)

- To find a projective mapping onto any simple $L(\lambda)$:
  - take sufficiently large n such that $\mu = \lambda + n\rho$ is dominant
  - $M(\mu) \otimes L(n\rho)$ is projective and has SF with top factor being $M(\mu - n\rho) = M(\lambda)$, hence has quotient $L(\lambda)$
- To find a projective mapping onto any module:
  - follows from a routine induction on composition length.

Projectives

Consequently,

Proposition (Projective covers)

Every $M \in \mathscr{O}$ has a **projective cover***.*
Those for $L(\lambda)$ are denoted $P(\lambda)$; these are the PIMs of $\mathscr{O}$ ($\overset{\sim}{\longleftrightarrow}$ simples).
They are also the proj. covers of $M(\lambda)$.

Proposition (Standard filtration of projectives)

$P(\lambda)$ (hence every projective in $\mathscr{O}$) has a standard filtration.

Proof sketch

- $P(\lambda)$ is a direct summand of the $M(\mu) \otimes L$ we constructed.
- Now any direct summand of a module N with SF, itself has SF
  - Take a maximal weight λ in N, consider the associated $M(\lambda) \to N$ and show using the SF that it is an embedding, then quotient by $M(\lambda)$ and induct

Projectives

Theorem (BGG Reciprocity)

$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$

Proof sketch

$$\begin{aligned}(P(\lambda) : M(\mu)) &= \dim \operatorname{Hom}(P(\lambda), M(\mu)^\vee) \quad (*)\\ &= [M(\mu)^\vee : L(\lambda)] \quad (\text{use exactness of } \operatorname{Hom}(P(\lambda), -))\\ &= [M(\mu) : L(\lambda)] \quad \square\end{aligned}$$

(*) is true with $P(\lambda)$ replaced with any M which has SF

- Induct on the SF, using long exact sequence for $\operatorname{Ext}(-, M(\mu)^\vee))$ and
  - $\dim \operatorname{Hom}(M(\lambda), M(\mu)^\vee) = \delta_{\lambda,\mu}$
  - $\operatorname{Ext}(M(\lambda), M(\mu)^\vee) = 0$

Remark

$M^\vee$ denotes the BGG *dual* in $\mathscr{O}$. Key properties: preserves formal characters and simple modules.

Interlude: Projective dimension

The natural next question to ask, after finding enough projectives in $\mathcal{O}$, is regarding projective resolutions and projective dimension.

In fact we have:

Theorem (Global dimension of $\mathcal{O}_0$)

$\mathcal{O}_0$ has global dimension precisely $2l(w_o) = 2|\Phi^+|$.

Proof sketch

1. Show that p.d.$M_w \leq l(w)$ and p.d.$L_w \leq 2l(w_o) - l(w)$
   - Inductively (on $l(w)$)
   - Use the filtration of projectives by Vermas and Vermas by simples
2. Show that p.d.$L(0) = 2l(w_o)$
   - show that $\mathrm{Ext}_{\mathcal{O}}^{2l(w_o)}(L(0), L(0))$ is non-zero
   - To do this we will need a short excursion into Ext and relative cohomology.

Ext and relative Lie algebra cohomology

Every module in $\mathscr{O}$ is a $(\mathfrak{g}, \mathfrak{h})$-module (in the sense of relative Lie algebra cohomology).

Theorem (Ext in $\mathscr{O}$ and relative Lie algebra cohomology)

There are canonical isomorphisms
$Ext^n_{\mathscr{O}}(M, N) \cong Ext^n_{(\mathfrak{g},\mathfrak{h})}(M, N)(\cong H^n(\mathfrak{g}, \mathfrak{h}, Hom_{\mathbb{C}}(M, N)))$

Theorem (Ext in $\mathcal{O}$ and relative Lie algebra cohomology)

There are canonical isomorphisms
$Ext^n_{\mathcal{O}}(M, N) \cong Ext^n_{(\mathfrak{g},\mathfrak{h})}(M, N)(\cong H^n(\mathfrak{g}, \mathfrak{h}, Hom_{\mathbb{C}}(M, N)))$

Proof sketch

- Every module in $\mathcal{O}$ is a $(\mathfrak{g}, \mathfrak{h})$-module:
  - By characterisation of Ext^n by "long exact extensions": $\exists$ a natural transformation $\mathrm{Ext}^n_{\mathcal{O}} \to \mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{h})}$
- Examining how this natural transformation maps between long exact sequence for both Exts on projective cover of any module:
  - See that it suffices show $\mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{h})}(P, N)$ vanishes for projectives $P \in \mathcal{O}$
- We have already constructed suitably general projectives $\in \mathcal{O}$ which are induced modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (-)$:
  - What remains conceptually are then computations in $\mathrm{Ext}_{(\mathfrak{g},\mathfrak{h})}$.

Reference

Delorme, "Extensions dans la categorie $\mathcal{O}$ de Bernstein–Gelfand–Gelfand" (it's in French, but math papers in French are surprisingly easy to read)

Projective dimension

Returning to global dimension:

Theorem (Global dimension of $\mathscr{O}_0$)

$\mathscr{O}_0$ has global dimension precisely $2l(w_o) = 2|\Phi^+|$.

Proof sketch ((cont.))

We have $\mathrm{Ext}_{\mathscr{O}}^{2l(w_o)}(L(0), L(0)) \cong H^{2l(w_o)}(\mathfrak{g}, \mathfrak{h}, \mathbb{C}) \cong H^{2l(w_o)}(G/B, \mathbb{C})$ and note $2l(w_o)$ is the ($\mathbb{R}$-)dimension of G/B.

4 Composition factors and embeddings of Vermas

Composition factor multiplicities

We return now to our natural question of the composition factor multiplicities.

In particular, BGG reciprocity reduces the problem of determining the Cartan matrix to finding the composition factor multiplicities of the Vermas (this itself would already be of independent interest).

Composition factor multiplicities

Natural first step: determine conditions under which $[M(\lambda) : L(\mu)] \neq 0$.

In fact we have:

Proposition (Composition factors and embeddings)

$[M(\lambda) : L(\mu)] \neq 0 \iff M(\mu) \subset M(\lambda)$.

Remark

This content of this result is **not** as straightforward as it seems: not all comp. factors $L(\mu)$ appear as part of embeddings of $M(\mu)$.
In fact Verma made an error regarding this in his pioneering thesis, and the correction was one of the motivations for BGG's work!

Embeddings of Vermas

Proposition (Composition factors and embeddings)

$[M(\lambda) : L(\mu)] \neq 0 \iff M(\mu) \subset M(\lambda).$

Proof sketch

- Induct on ht $(\lambda - \mu)$.
- Consider a comp. series of $M(\lambda)$; starting from $L(\mu)$ take the maximal subquotient of $M(\lambda)$ admitting a non-zero hom from $M(\mu)$.
- Next factor $L(\nu)$ in the comp. series must satisfy $\mathrm{Ext}(M(\mu), L(\nu)) \neq 0$; hence also $(\lambda >)\nu > \mu$.
- Then use the SF of $P(\mu)$ and the non-split extension to show that $(P(\mu) : M(\nu)) \neq 0$; hence (BGG reciprocity) $[M(\nu) : L(\mu)] \neq 0$.

Reference

Moody-Pianzola, "Lie Algebras with Triangular Decompositions", 2.11.
(Our proof here is simpler; they work in generality of Kac-Moody algebras.)

Embeddings of Vermas

Consequence:
to find if composition factor appears, just need to look at embeddings of Vermas.

Recall when showing $\mathcal{O}_\lambda$ indecomposable:

Recall

λ integral, s simple, $s \cdot \lambda < \lambda$. Then $M(s \cdot \lambda) \subset M(\lambda)$.

Embeddings of Vermas

The key to the general case is to remove the requirement that s be simple.

Proposition (Verma)

λ integral, s a **reflection**, $s \cdot \lambda < \lambda$. *Then* $M(s \cdot \lambda) \subset M(\lambda)$.

Proof sketch

- Basic idea: relative positions of Vermas with respect to a **simple** s can be pinned down via computations.
- A (reduced) sequence of **simples** brings $s \cdot \lambda$ to λ^+ (dominant);
- Look at the same sequence of **simples** applied to λ;
- And compare the relative positions of the two.

Overall it is really using s **simple** + technicalities. We refer to:

Reference

Dixmier, "Enveloping algebras", 7.6.11

BGG Theorem

The main celebrated result is essentially the converse.

Theorem (BGG Theorem)

Every embedding $M(\mu) \subset M(\lambda)$ is a composition of embeddings of the preceding form ($M(s \cdot \lambda) \subset M(\lambda)$, $s \cdot \lambda < \lambda$, s a ***reflection****).*

In $\mathscr{O}_0$, this may be stated as:

Theorem (BGG Theorem in $\mathscr{O}_0$)

$M_w \subset M_{w'} \iff w < w'$ *(Bruhat order on W)*

Remark

Here we follow the BGG convention for the Bruhat ordering - id is the *largest* element - which may be the reverse of more common conventions for the Bruhat order.

Theorem (BGG Theorem in $\mathscr{O}_0$)

$M_w \subset M_{w'} \iff w < w'$ *(Bruhat order on W)*

Proof sketch

For the integral case, in $\mathscr{O}_0$: Suppose $M_w \subset M_{w'}$, WTP $w < w'$.

- Again basic idea: pin down relative position of Vermas wrt simple s.
  - Here we use additional ideas in Verma's original thesis (only for integral weights) to exploit symmetry after applying w_o (longest element).
- Induct on $l(w)$:
  - Choose a simple s with $sw > w$: show that $M_w \subset M_{sw'}$.
    - If $sw' > w'$, then show that $M_{sw} \subset M_{sw'}$;
    - If $sw' \leq w'$, then show that $M_{sw} \subset M_{w'}$;

Reference

For the integral case: van den Hombergh, "Note on a paper by Bernstein, Gelfand and Gelfand on Verma modules".
Original approach by BGG for general weights: Dixmier, "Enveloping algebras", 7.6.23 (highly involved!).

Digression: Weyl character formula

Now we know more about embeddings of Vermas, we can return to WCF.

Recall (Weyl character formula in BGG context)

For λ dominant integral, $[L(\lambda)] = \sum_{w \in W} (-1)^{l(w)} [M(w \cdot \lambda)]$

With embeddings of Vermas: natural to expect that there is an actual homological realisation of this formula.

Furthermore, the formula resembles an alternating sum, hence once might expect it to come from the Euler characteristic of an exact sequence.

BGG resolution

Theorem (BGG resolution)

For dominant integral λ there is an exact sequence

$$0 \to C_{l(w_o)} \to C_{l(w_o)-1} \to \cdots \to C_1 \to C_0 \to L(\lambda) \to 0$$

with $C_i = \oplus_{w \in W, l(w)=i} M_w$.
The maps are precisely the possible embeddings of Vermas $M_w \hookrightarrow M_{w'}$, suitably scaled by scalars in $\mathbb{C}$ (in fact, ± 1) so that we obtain a complex.

Proof sketch

We may do everything for $\lambda = 0$, then apply the translation functors T_0^λ. Note $L(0)$ is the trivial module.

The main highlight is that the BGG resolution is first constructed using the standard resolution in (relative) Lie algebra cohomology for $(\mathfrak{g}, \mathfrak{b})$.

Theorem (BGG resolution)

$$0 \to C_{l(w_o)} \to C_{l(w_o)-1} \to \cdots \to C_1 \to C_0 \to L(\lambda) \to 0$$

with $C_i = \oplus_{w \in W, l(w)=i} M_w$.

Proof sketch

- Set $C_k = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \wedge^k(\mathfrak{g}/\mathfrak{b})$;
  - since $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ we have a natural identification with the standard resolution for $\mathfrak{n}^-$.
- Weights of $\wedge^k(\mathfrak{g}/\mathfrak{b})$ are negative sums of k distinct positive roots
  - Note $w \cdot 0 = w\rho - \rho$ is a negative sum of k distinct positive roots for $l(w) = k$, so we get the correct weights (to get M_w)
- Projecting everything onto $\mathscr{O}_0$ one shows the resolution is as desired.

Reference

Rocha–Caridi, "Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Bernstein–Gelfand–Gelfand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module", 10.5

BGG resolution

Remark (BGG resolution and cohomology)

Since the BGG resolution is a resolution by free $U(\mathfrak{n}^-)$-modules, it naturally provides a relatively easy way to compute the $\mathfrak{n}^-$-cohomologies (then transferring everything to $\mathfrak{n}$ by symmetry)

$$H^i(\mathfrak{n}, L(\lambda)) \cong \oplus_{w \in W, l(w)=i} \mathbb{C}_{w \cdot \lambda} \quad \text{as } \mathfrak{h}\text{-modules;}$$

this is known as Bott's/Kostant's theorem.

Bott's/Kostant's theorem can be used to give a 'short' (after some work regarding sheaf cohomology) proof of the Borel-Weil-Bott theorem (generalisation of Borel-Weil theorem in higher cohomologies).

Kazhdan-Lusztig 'conjecture'

Now we know all the composition factors of M_w are $L_{w'}$ with $w' \leq w$.

The final natural question is to determine the multiplicities.

It turns out this is an extremely difficult problem and is one of the main reasons for the significance of $\mathscr{O}$.

Theorem (K-L Conjecture, sketch)

Kazhdan-Lusztig polynomials: express change-of-basis relation between the standard basis and a special "K-L basis" of the Hecke algebra $H(W)$ (a deformation of the group algebra $\mathbb{Z}[W]$).

K-L Conjecture (now theorem): composition factor multiplicities are given precisely by the Kazhdan-Lusztig polynomials.

Remark

The WCF (in BGG form) is one (very) special case of the K-L conjecture.

Kazhdan-Lusztig 'conjecture'

Proof sketch

- Original proofs by Beilinson-Bernstein, Brylinski-Kashiwara in the 80s: extremely involved, uses geometry of flag variety G/B and much more
- 'Modern' approach starting with Soergel in the 90s:
  - $\mathscr{O}_0$ with suitable translation functors categorifies $\mathbb{Z}[W]$
    - Here projectives and BGG reciprocity plays a central role: roughly speaking, PIMs $\overset{\sim}{\longleftrightarrow}$ K-L basis; Vermas $\overset{\sim}{\longleftrightarrow}$ std basis.
  - Soergel introduced *Soergel bimodules* which categorify $H(W)$
  - Allows us to transfer the problem to the setting of Soergel bimodules, which have geometric interpretation

Summary

Theorem (Summary)

1. $\mathscr{O} = \oplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathscr{O}_\lambda$;
   $\mathscr{O}_\lambda$ is indecomposable for λ integral
2. *$\mathscr{O}_\lambda$ is equivalent to $\mathscr{O}_0$ for λ dominant integral;*
   $\mathscr{O}_0$ has only $|W|$ many simples $L_w := L(w \cdot 0)$ and corresponding Vermas $M_w := M(w \cdot 0)$ (and Grothendieck group of finite rank $|W|$).
3. *$P(\lambda) =$ projective cover of $L(\lambda)$ (and $M(\lambda)$);*
   BGG reciprocity: $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$;
   $\mathscr{O}_0$ has global dimension precisely $2l(w_o) = 2|\Phi^+|$.
4. *BGG theorem: In $\mathscr{O}_0$:*
   $[M_{w'} : L_w] \neq 0 \iff M_w \subset M_{w'} \iff w < w'$ *in the Bruhat order;*
   K-L 'conjecture': composition factor multiplicities are given by 'change-of-basis' in the Hecke algebra.
5. *BGG resolution 'realises' WCF as its Euler characteristic via embeddings of Vermas.*

Looking ahead...

On Lie groups:

- There is analogous highest weight theory for compact Lie groups K (wrt a maximal torus T); this is in the same spirit as how we viewed mods in $\mathscr{O}$ as $(\mathfrak{g}, \mathfrak{h})$-mods.

  What about non-compact (reductive) Lie groups G?
  One of the motivations for $\mathscr{O}$ was in fact to study the Harish-Chandra $(\mathfrak{g}, K)$-modules, K a maximal compact subgroup of the real G.
  There are in fact close parallels of this theory with that of category $\mathscr{O}$.

Looking ahead...

Some extensions of $\mathscr{O}$:

- Parabolic version of $\mathscr{O}$, $\mathscr{O}_{\mathfrak{p}}$:
  - Levi $\mathfrak{l}$ plays the role of $\mathfrak{h}$
  - Nilradical $\mathfrak{u}$ plays the role of $\mathfrak{n}$
  - $\exists$ analogues of most of the results for $\mathscr{O}$ in $\mathscr{O}_{\mathfrak{p}}$.
- (Non-trivial) extension of $\mathscr{O}$ to Kac-Moody algebras:
  - We really mainly used the triangular decomposition of $\mathfrak{g}$;
  - Kac-Moody algebras have a similar decomposition and analogous notion of highest weight and Verma modules.

and much more...

Thank you!

On duality

$M^\vee$ denotes the **dual** of M in $\mathcal{O}$. We do not strictly need the duality in $\mathcal{O}$ subsequently, but it is nonetheless a valuable construction, especially from a homological viewpoint (injectives, Ext).

Recall (Usual duality)

The vector space dual M^* carries a **right** $\mathfrak{g}$ action by $(f \cdot x)(v) = f(x \cdot v)$.
In the usual case, we transfer this to a *left* $\mathfrak{g}$-action by the anti-involution $x \mapsto -x$.
However, this negates the weights, and so is not desirable for analysing weight spaces / composition factors.
More importantly, when M is infinite-dimensional, M^* is not well-behaved.

On duality

So, we do two things:

Definition (BGG dual)

1. Recall the weight spaces of M are f.d.. So we may consider taking the duals of weight spaces of M, in other words, the weight vectors in M^*.
2. Transfer the action using a *Chevalley anti-involution* τ which fixes $\mathfrak{h}$ pointwise (hence sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$).

The result is the dual module $M^\vee$ we are looking for.

Proposition (Properties of duality)

The duality is exact, contravariant, involutory, preserves formal characters and simple modules. (Hence it is also called the simple-preserving duality, in contrast with the usual duality.)

Reference

Humphreys 14.3 for the Chevalley anti-involution.

On duality

Remark

Duality is most often applied in two settings:

- $M(\lambda)^\vee$: the dual of Verma has unique simple submodule (socle) $L(\lambda)$, all weights $\leq \lambda$, and hence has quite a restricted structure (wrt highest weight modules)
- $P(\lambda)^\vee$: the indecomposable injectives.