1. Motivation on algebraic groups.

G connected ~~semisimple~~ (reductive) algebraic group.

Def^{n}/Prop.: closed subgp. P parabolic $\iff$ G/P projective.

$\iff$ P contains Borel subgp. (maximal closed, connected solvable).

Pf: (Borel's fixed pt. thm.).

~~Def^{n}~~: (why interesting?)

Rmk: classical G: $G/B \to$ full flag variety

$G/P \to$ partial flag variety,

Parabolic subalgs.

Def^{n}: Parabolic subalgebra $\iff$ contains Borel subalgebra.

Notat^{n}: $\mathfrak{b}$ Borel, $\mathfrak{h}$ Cartan, Φ root system, nilradical $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Δ simple system 6 Φ^{+} positive roots.

Thm: $\left\{\begin{matrix}\text{subsets of vertices} \\ \text{of Dynkin diagram}\end{matrix}\right\}$.

$$\left\{\begin{matrix}\text{subsets} \\ \text{of } \Delta\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Parabolic} \\ \mathfrak{p} \supset \mathfrak{b}\end{matrix}\right\} \xhookrightarrow{\text{(classical types)}} \left\{\begin{matrix}\text{partial} \\ \text{flag varieties}\end{matrix}\right\}$$

$\left\{\begin{matrix}\text{open faces} \\ \text{of Weyl} \\ \text{chambers}\end{matrix}\right\}$

Pf: (1) Given subset $\Theta \subset \Delta$, let $\langle\Theta\rangle$ denote root system gen. by Θ and $\mathfrak{p}_\Theta := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^{+} \cup \langle\Theta\rangle} \mathfrak{g}_\alpha$ is parabolic.

~~Pf~~: ~~(1)~~ $\mathfrak{p} \supset \mathfrak{h}$ so has root space decomp.

(2) Every root space is 1-dimensional.

Write $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$.

~~(3)~~ (a) Given neg. root $\in \Gamma$, write as sum of simple roots, then use that all positive roots are in Γ to show these simple roots in Γ.

(b) Given simple roots with negative in Γ, any neg. root in the system gen. by them is neg. integer combinat^{n} of them, hence must be in Γ. □

Prob is: if $\alpha = \sum \alpha_1 + \dots + \alpha_n$, not all subsets of $\{\alpha_i\}$ have sum being root!
Lem: $\exists \alpha_i$ w/ α_i, $\sum \alpha_i$ being root.

~~Lem~~. ~~Cor~~. ~~Cor~~. Lem. Any parabolic $\mathfrak{p}_\Theta$ has reductive Levi decomp. (caution: differs from 'usual' Levi decomp = ss $\oplus$ solvable radical!)

$$\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$$
$$= \left(\mathfrak{h} \oplus \bigoplus_{\alpha \in \langle\Theta\rangle} \mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha \in \Phi^+ \setminus \langle\Theta\rangle^+} \mathfrak{g}_\alpha\right)$$

$\mathfrak{n}_\Theta$ is nilradical

$\mathfrak{l}_\Theta$ is reductive & called Levi subalg.

Pf: std. verificat^{n} using root system.

Rmk: Levi is reductive. So Levi of Levi is still Levi (subset of subset of simple roots, modulo certain conjugat^{n}s)

Conjugacy of parabolic/Levi subalg.

Lem:
(a) $\mathfrak{p}_{\Theta_1}$ G_{ad}-conj. to $\mathfrak{p}_{\Theta_2}$ $\Leftrightarrow$ $\Theta_1 = \Theta_2$.
(b) $\mathfrak{l}_{\Theta_1}$ G_{ad}-conj. to $\mathfrak{l}_{\Theta_2}$ $\Leftrightarrow$ $\langle\Theta_1\rangle$, $\langle\Theta_2\rangle$ W-conj.

Rmk/. Example: $\mathfrak{sl}_3$: $\bullet\!\!-\!\!\circ$ $\quad$ $\circ\!\!-\!\!\bullet$ } graph automorphism

$$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ & & \times \end{pmatrix} \quad \begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & \times & \times \end{pmatrix}$$
} conj. by outer automorphism ($A \mapsto -JA^TJ^{-1}$, $J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, negative transpose along anti-diagonal)

Pf: key: recall $W \cong N_{G_{ad}}(\mathfrak{h}) / C_{G_{ad}}(\mathfrak{h})$ naturally.

(a)
$$\begin{array}{ccc} \mathfrak{b} & \longmapsto & \mathfrak{b}' \quad \mathfrak{b}' \overset{\sim}{\leftarrow} \mathfrak{b} \text{ via } P_{2ad}. \\ \cap & & \cap \\ \mathfrak{p}_{\Theta_1} & \xrightarrow[\sigma]{\in G_{ad}} & \mathfrak{p}_{\Theta_2} \end{array}$$

May assume $\sigma(\mathfrak{b}) = \mathfrak{b}$.
$\sigma(\mathfrak{h}) = \mathfrak{h}$ $\longrightarrow$ then σ as element of W preserves ~~pos~~ Φ^+ so must send Δ to itself i.e. is identity in W & on $\mathfrak{h}$.
So also preserves -ve root spaces.

(b) Similar, may assume $\sigma(\mathfrak{h}) = \mathfrak{h}$ hence given by a w, which hence conjugates $\langle\Theta_1\rangle$, $\langle\Theta_2\rangle$. $\square$

Converse again use $W \cong N/C$.

Example 1. $\mathfrak{g} = \mathfrak{sl}_n$, $W \cong S_n$ acts naturally by ~~permuting~~ diagonal elements. ~~of permuting size~~

Therefore Any Levi subalg. G_{ad}-conjugate to one of form

$$\begin{array}{c} \\ d_1 \\ d_2 \\ \\ \\ d_k \end{array} \begin{array}{c} d_1 \quad d_2 \quad \cdots \quad d_k \\ \begin{pmatrix} \boxed{} & & & \\ & \boxed{} & & \\ & & \ddots & \\ & & & \boxed{} \end{pmatrix} \end{array} \qquad (d_1 \geq \dots \geq d_k \geq 1, \; d_{k+1} = \dots = d_n = 0)$$

i.e. correspond to a partition d of n, & corresponding parabolic is denoted $p(d)$.

Example 2: Jacobson-Morozov parabolics $\boxed{\text{GOTO pg 4}}$

Levi / ~~toral~~ (for Bala-Carter):

(2) Recall earlier: $\mathfrak{l}_\Theta = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_\alpha$ reductive. } Proof that they're reductive similar!

swap (1) Recall: X semisimple, $\mathfrak{g}^X = \mathfrak{h} \oplus \bigoplus_{\alpha(X) = 0} \mathfrak{g}_\alpha$ reductive.

Not a coincidence!

~~Def~~ⁿ: Toral subalg. consist of semisimple elements (of $\mathfrak{g}$).

Thm: X nilpotent.

$$\left\{ \begin{array}{c} \text{minimal Levi} \\ \text{containing } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{maximal toral} \\ \text{~~containing~~ in } \mathfrak{g}^X \end{array} \right\}.$$

Lem: $\mathfrak{t}$ toral $\Rightarrow \mathfrak{g}^{\mathfrak{t}}$ Levi.

Pf: $\mathfrak{g}^{\mathfrak{t}} = \mathfrak{h} \oplus \bigoplus_{\alpha \text{ kills } \mathfrak{t}} \mathfrak{g}_\alpha$ $\longrightarrow$ check this is subroot system w/ base being subset of another.

Lem: $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_\alpha$ Levi with center $\mathfrak{c} = \bigcap_{\alpha \in \langle \Theta \rangle} \ker \alpha \subseteq \mathfrak{h}$. (Just note center, in fact $\mathfrak{g}^{\mathfrak{l}} \subset \mathfrak{h}$.)

Then clearly $\mathfrak{c} \subset \mathfrak{h}$ toral

and $\mathfrak{g}^{\mathfrak{c}} = \mathfrak{l}$. (linear algebra: if root $\notin$ span of $\langle \Theta \rangle$ in $\mathfrak{h}^*$ then cannot kill all $\mathfrak{c}$)

Consequence: in thm. above,

$$\mathfrak{l} \longmapsto z(\mathfrak{l}) = \mathfrak{g}^{\mathfrak{l}} = \mathfrak{c}$$

$$\mathfrak{g}^{\mathfrak{t}} \longleftarrow\!\shortmid \; \mathfrak{t}$$

$\square$

Cor: Any two minimal Levi containing X are G_{ad}^X-conj.

~~Prop~~: (Not complete, but)

Pf idea: cf. conjugacy of maximal toral subalg. in $\mathfrak{g}^X$. $\square$

Jacobson-Morozov parabolics

Recall: $\{H, X, Y\}$ sl_2-triple,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \quad \text{(weight spaces for ad } H\text{)}$$

~~Define:~~

Prop: $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ is parabolic

w $\mathfrak{l} = \mathfrak{g}_0$, $\mathfrak{n} = \bigoplus_{i > 0} \mathfrak{g}_i$.

Pf: Recall ~~may~~ wlog $H \in \mathfrak{h}$ Cartan

w $\alpha(H) = 0, 1, 2$ (after conjugating the ~~to simple~~ root system by W i.e. G_{ad})
$\forall \alpha \in \Delta$.

Take $\Theta = \{\alpha \mid \alpha(H) = 0\}$.

Rest is immediate verificat.

Lem: ~~Rmk~~: $\mathfrak{p}$ is uniquely determined by X. Pf: by Kostant, any two sl_2-triples w same X are G_{ad}^X-conj., hence the assoc. $\mathfrak{p}$ are G_{ad}^X-conj., but $\mathfrak{g}^X \subset \mathfrak{p}$
$\qquad \mathfrak{g}^X = \bigoplus_{i \geq 0} \mathfrak{g}_i^X$.

Rmk: also Isaac ~~showed~~ noted

$$\mathfrak{g}^X = \underbrace{\mathfrak{g}_0^X}_{\mathfrak{g}^\phi \text{ (}sl_2\text{-theory)}} \oplus \mathfrak{n}^X.$$

Rmk: $\to \mathfrak{g}_0$ is centraliser of H! ~~(GOTO pg 3)~~.

$(\mathfrak{g}_0 = \mathfrak{g}^H)$.

Rmk: $\to$ In above, taking α s.t. α kills H, motivates Levi/toral!

$\boxed{\text{GOTO pg 3}}$

Focus on: why impt for any two minimal Levi to be conjugate?

Recall: Idea: want to reduce study of nilpotent orbits to that of smaller, ideally also reductive (semisimple) subalgebras, the smaller the better.

[Naturally: look at minimal Levi subalg. containing X]
(smallest)

For a good theory, expect some ~~biject. to the~~ correspondence to nilpotent orbits in l which shouldn't depend on choice of l. * So, if any two minimal Levi conjugates then to an orbit O_X we can attach minimal Levi without fear!

Thus: (Bala-Carter)

$$\left\{\begin{array}{c}\text{nilpotent orbits}\\ \text{of } \mathfrak{g}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}G_{ad}\text{-conjugacy classes}\\ \text{of } (l, p_l)\ \cancel{\text{with } X}\\ \cancel{\text{distinguished in } l,}\end{array}\right\}$$

[~~i.e.~~ p_l is ~~distinguished~~ Jacobson-Morozov parabolic subalg. corresponding to distinguished X]

$O_X \longleftrightarrow$ (minimal l, Jacobson-Morozov parabolic p_l of X)

Rmk. Distinguished orbit O_X in l $\iff$ l is the only Levi of l containg X.

Def^n: Visibly, just means l is minimal Levi.

*GOTO rmks (1), (2), (3)

Key pts: Well-definedness. Any X uniquely determines Jacobson-Morozov parabolic. Then any two minimal Levi containg X are G_{ad}-conjugate!

Inverse: ~~Need to produce~~ To go from l to $\mathfrak{g}$ turns out to be naive (just take $O_{\mathfrak{g}} = G_{ad}\cdot O_l$).
The key point is how to go from parabolic to a nilpotent orbit?

(1) Rmk: Really, two things going on here:
(1) {nilp. orbit in $\mathfrak{g}$} $\leftrightarrow$ {G_{ad}-conj. classes of distinguished orbits of Levi}
(2) {distinguished orbits} $\leftrightarrow$ {distinguished parabolic subalgebra (which must be J-M parabolic subalg.)}

Rmk (2): Other nice thing: We have seen G_{ad}-conj. of l $\longleftrightarrow$ W-conj. of subset of simple roots. So we have natural parametrizat^n by taking the first subset in lexicographical order, [also sometimes called "standard Gamma"].

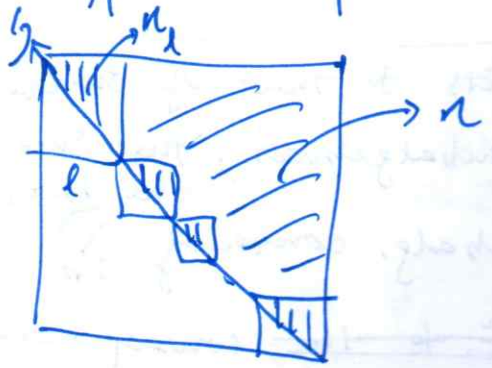
Induced nilpotent orbits:

$\mathfrak{g}$ with parabolic $p = l \oplus n$. Given nilpotent orbit O_l in l, how to pass to new orbit $O_{\mathfrak{g}}$? induced

(prototypical example: $O_l = \{0\}$, in which case called Richardson orbit.)

Rmk (3): Again, we also know parabolic $\longleftrightarrow$ subset of simple roots (up to conj.)
So parametrizat^n of Bala-Carter is just subset of subset of simple roots, which can be done e.g. on Dynkin diagram.

Prototypical example: $\mathfrak{sl}_n$.



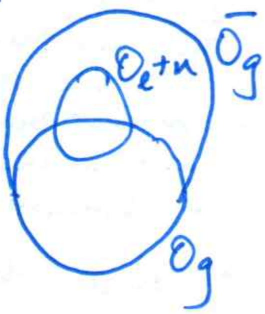
 $= \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$

~~Obvious first point: $\mathcal{O}_\mathfrak{l}$ is a priori only nilpotent in $\mathfrak{l}$.~~

Thm: for any nilpotent orbit $\mathcal{O}_\mathfrak{l}$,
$\exists$ unique orbit $\mathcal{O}_\mathfrak{g}$ of max. dimension meeting $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ in open dense set.
Denote $\mathcal{O}_\mathfrak{g} := \mathrm{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$.

Pf: (1) Obvious first point: $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ nilpotent?
Given $y \in \mathcal{O}_\mathfrak{l}$, conjugate it via L_{ad} to $\mathfrak{n}_\mathfrak{l} \subset \mathfrak{n}^+$.
Also $\mathfrak{n}$ is ideal in $\mathfrak{p}$ so L_{ad} stabilises $\mathfrak{n}$.
$\Rightarrow \mathcal{O}_\mathfrak{l} + \mathfrak{n} \subset \mathcal{N} = \{$nilpotent elements in $\mathfrak{g}\}$.
(2) $\mathcal{O}_\mathfrak{l}$ irred (as L_{ad} connected) & $\mathfrak{n}$ irred $\Rightarrow \mathcal{O}_\mathfrak{l} + \mathfrak{n}$ irred.
(3) Finitely many nilpotent orbits in $\mathfrak{g}$, say
$\mathcal{O}_1, \dots, \mathcal{O}_k$.
Consider $\overline{\mathcal{O}_1 \cap (\mathcal{O}_\mathfrak{l} + \mathfrak{n})}, \dots, \overline{\mathcal{O}_k \cap (\mathcal{O}_\mathfrak{l} + \mathfrak{n})}$ closed in union $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$.
By irreducibility, one of them is the whole $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$.
Denote this by $\mathcal{O}_\mathfrak{g}$. Already it meets $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ in dense set.

(4)



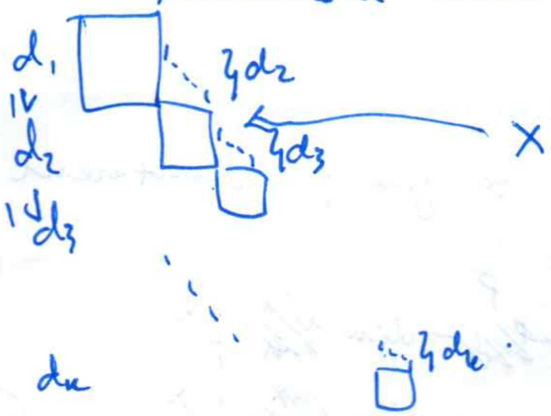
Recall boundary of $\mathcal{O}_\mathfrak{g}$ is union of orbits of smaller dimension.
So uniqueness + max dim. follows.
Also $\mathcal{O}_\mathfrak{g}$ open in $\overline{\mathcal{O}_\mathfrak{g}}$, so openness follows.

Cor. $\mathcal{O}_\mathfrak{g}$ dominates (in partial order) all other orbits meeting $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$.

Example: Producing nilpotent orbit from parabolic in $\mathfrak{g} = \mathfrak{sl}_n$.

Recall every Levi conjugate to that of $\mathfrak{p}(d)$ for partition d.

Richardson orbit: $\mathcal{O}_{\mathfrak{l}} = \{0\}$. Just consider $\mathfrak{n}(d)$ (Staircase)



Clearly

rank $X = d_2 + \ldots$

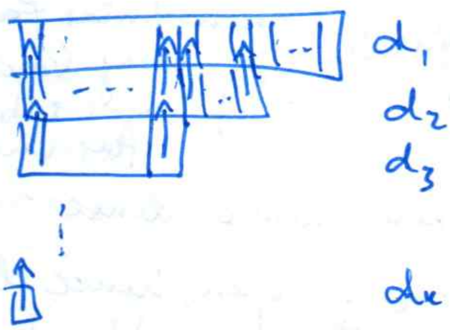
rank $X^2 = d_3 + \cdots$

$\vdots$

OTOH for any other nilpotent Y, clearly rank $Y^i \leq d_{i+1} + \cdots = \text{rank } X^i$ as all $d_1, \ldots, d_i$ subspaces are killed.

So by result from Isaac's talk, $\mathcal{O}_X \geq \mathcal{O}_Y$ dominates all other nilpotent orbits meeting $\mathfrak{n}(d)$ & hence $\mathcal{O}_X$ is the desired Richardson orbit.

~~In X's JCF, there are $(d_1 - d_2)$ many blocks of size 1,~~



Associate to each cell of the Young tableaux the corresponding std. basis element. Then X acts as the arrows shown. So clearly its JCF has partition corresponding to transpose d^t!

Thm. Partition of $\mathrm{Ind}_{\mathfrak{p}(d)}^{\mathfrak{sl}_n}(\mathcal{O}_0) = d^t$.

Every orbit is a Richardson orbit.

Recall now ~~dim~~ in $\mathfrak{sl}_n$ $\dim \mathfrak{g}^x = \sum s_i^2 - 1$ where s_i are the elements of the transpose of x's partition.

$\therefore$ In the above, $\dim \mathfrak{g}^x = \sum d_i^2 - 1$.

But $\sum d_i^2 - 1 = \dim \mathfrak{l}(d) = \dim(\text{stabiliser of } \mathcal{O}_0 \text{ in } \mathfrak{l})$!

Illustrates the following dimension formula:

Thm: $y \in \mathcal{O}_l$ and $X \in \mathcal{O}_g = \mathrm{Ind}_p^g(\mathcal{O}_l)$.

(Dimension formula) Then $\dim l^y = \dim g^X$,

equiv. $\mathrm{codim}_l(\mathcal{O}_l) = \mathrm{codim}_g(\mathcal{O}_g)$,

equiv. $\dim \mathcal{O}_g = \cancel{2}\dim \mathcal{O}_l + 2\dim n$.

Pf <u>sketch</u>: Full pf. is too technical to give full treatment.

① Take any $X \in \mathcal{O}_g \cap (\mathcal{O}_l + n)$.

$$\dim(\underbrace{\cancel{P_{ad}\text{-orbit through } X}}_{P_{ad}\cdot X}) = \dim \cancel{P_{ad}}\, p - \dim \cancel{P_{ad}}\, p^X$$
$$\geq \dim p - \dim \cancel{P_{ad}}\, g^X \quad \text{— ⓐ}$$

OTOH $\dim(P_{ad}\cdot X) \leq \dim(\mathcal{O}_l + n)$ (note P_{ad} stabilises ~~$\mathcal{O}_l$~~ n and N_{ad} sends $y \in \mathcal{O}_l$ to $y + n$ ~~⊇~~, all b/c n is ideal) ⓑ

$$= \dim l - \dim l^y + \dim n$$
$$= \dim p - \dim l^y$$

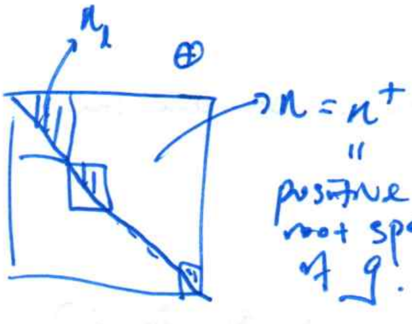
ⓐ, ⓑ $\Rightarrow \dim l^y \leq \dim g^X$.

② OTOH: *assuming $\dim(\mathcal{O}_X \cap n^+) = \frac{1}{2}\dim \mathcal{O}_X$ in g. (non-trivial fact; pf. using variety of Borel subgroups/flag variety)

$\mathcal{O}_g$ meets $y+n$ in open, hence dense.

Apply L_{ad}: $\mathcal{O}_g$ meets $y'+n$ in open, hence dense $\forall\, y' \in \mathcal{O}_l$.

$\Rightarrow \mathcal{O}_g$ meets $(\mathcal{O}_l \cap n_l) + n$ in open dense



$$\tfrac{1}{2}\dim \mathcal{O}_X = \dim(\mathcal{O}_X \cap n^+)$$
$$\geq \dim\big(\mathcal{O}_X \cap ((\mathcal{O}_l \cap n_l) + n)\big)$$
$$= \dim\big((\mathcal{O}_l \cap n_l) + n\big)$$
$$= \cancel{\dim}\ \tfrac{1}{2}\dim \mathcal{O}_l + \dim n$$

(① $\mathcal{O}_l \cap n_l$ has all irred. comp. same dim! ② locally closed subset of affine has same dim as its closure ③ or use $\mathbb{F}_2$ field !?)

$\Rightarrow \dim l^y \geq \dim g^X$.

<u>Cor.</u> In ① above, $\dim(P_{ad}\cdot X) = \dim(\mathcal{O}_l + n)$

From this deduce $\mathcal{O}_g \cap (\mathcal{O}_l + n)$ is a single P_{ad}-orbit.

Recall Levi ∩ Levi is Levi.

Cor $\mathfrak{p}_1, \mathfrak{p}_2$ with $\mathfrak{l}_1 \subset \mathfrak{l}_2$.

(Transitivity of induct^{n}): Then

$$\mathrm{Ind}_{\mathfrak{p}_2}^{\mathfrak{g}}\left(\mathrm{Ind}_{\mathfrak{p}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})\right) = \mathrm{Ind}_{\mathfrak{p}_1}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_1}).$$

Pf. Both sides clearly meet $\mathcal{O}_{\mathfrak{l}_1} + \mathfrak{n}_1$ by definit^{n}
and have same dimension by dimension formula (codim version) ∎

Expect induct^{n} to depend only on $\mathfrak{l}$, the Levi factor.

Thm: $\overline{\mathcal{O}_{\mathfrak{g}}} = \mathcal{N} \cap G_{ad} \cdot \overline{(Z(\mathfrak{l}) + \mathcal{O}_{\mathfrak{l}})}$, and $\mathcal{O}_{\mathfrak{g}}$ is the unique orbit of max dimension.
So $\mathcal{O}_{\mathfrak{g}}$ depends only on $\mathfrak{l}$, in particular, we may write $\mathcal{O}_{\mathfrak{g}} = \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}})$

Pf: Omitted for technicality (pf in Lusztig/Spaltenstein is few pages!) ∎
& reduces to casework for classical & exceptional types!
(Although uses Jacobson–Morozov)

$\boxed{\text{GOTO pg 12}}$ ($\mathfrak{sl}_n$ computat^{n}).

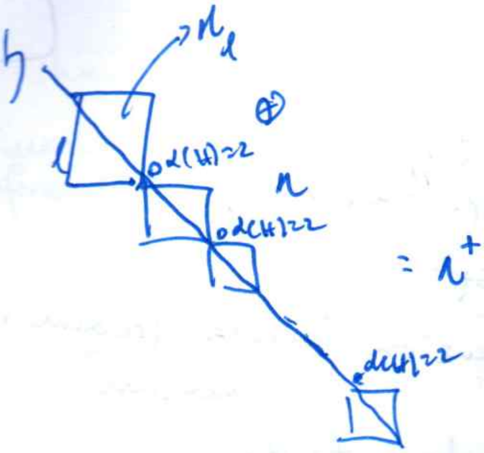
Weighted Dynkin & Induced orbits

Now: Seen e.g. of computat^{n} w partitions.
Recall: Parabolic ⟷ subset of nodes of Dynkin diagram.
Very natural qn:

How are induced orbits related to Dynkin diagram?

Thm: If WDD of $\mathcal{O}_{\mathfrak{g}}$ has vertices $v_1, \ldots, v_t$ labelled 2 $(\alpha_1, \ldots \alpha_t \in \Delta)$
(existence') then set $\Theta = \Delta \setminus \{\alpha_1, \ldots \alpha_t\}$
Then $\mathcal{O}_{\mathfrak{g}}$ is induced orbit from $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$.

Pf: Rmk that once we know Kostant's lemma, the idea is very intuitive. Proof given in the book is very cryptic!
Recall: (Kostant's lemma): $\mathcal{O}_x \cap \mathfrak{g}_2$ (open) dense in $\mathfrak{g}_2$.



① ~~Conjugate~~ $\exists$ rep. X in n^+
$= \underset{n_l}{\underset{\cap}{y}} + \underset{n}{\underset{\cap}{z}}$.

② n essentially consists of the non-removed simple roots $\alpha_1 \dots \alpha_t$ $(H)=2$

So by def^n $n \subset q_2$.

Also $X \in q_2$ so $y \in q_2$

③ Kostant: $y + n \subset q_2 \subset \bar{O}_X$

$n, \bar{O}_X$ are L ad-stable

so $O_y + n \subset \bar{O}_X$ as desired. □

This thm. does not say anything about what O_y looks like. Issue is $\{H, y, ?\}$ may not form sl_2-triple in l. Worse still, remaining diagram may not even be that of nilp. orbit in l! ~~In which case we can forget abt. H even being a neutral element~~.

Example: sl_5 $O_{[4,1]}$ $H = \begin{pmatrix} 3 &&&& \\ & 1 &&& \\ && -1 && \\ &&& -3 & \\ &&&& 0 \end{pmatrix}$ $\bar{H} = \begin{pmatrix} 3 &&&& \\ & 1 &&& \\ && 0 && \\ &&& -1 & \\ &&&& -3 \end{pmatrix}$ $\bar{X} = \begin{pmatrix} 0 & 1 &&& \\ & 0 & 0 & 1 & \\ && 0 & 0 & \\ &&& 0 & 1 \\ &&&& 0 \end{pmatrix}$.

If remove last vertex,

is <u>not</u> ~~Dy~~ WDD of nilp. orbit in sl_4!

$\bar{H} = \left(\begin{array}{cccc|c} 3 &&&& \\ & 1 &&& \\ && 0 && \\ &&& -1 & \\ \hline &&&& -3 \end{array}\right)$ $y = \left(\begin{array}{cccc|c} 0 & 1 &&& \\ & 0 & 0 & 1 & \\ && 0 & 0 & \\ &&& 0 & \\ \hline &&&& 0 \end{array}\right)$

y is nilpotent in l ~~[illegible]~~, $[\bar{H}, y] = 2y$ in l, but $\bar{H}, y$ do not form std. triple, for the extremely simple reason that $\mathrm{tr}\,\bar{H}$ (in the sl_4 part) $\neq 0$!

(i.e. $\bar{H}$ is not in the s.s. part of l)

Natural to assume then:

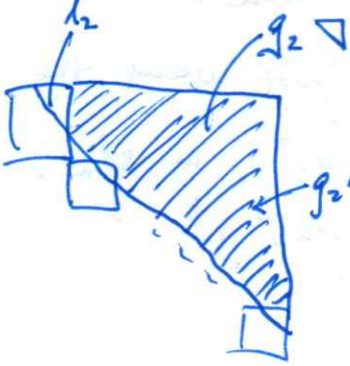
Thm: $\boxed{\text{If}}$ remaining diagram D' is diagram of O'_l in l,
('construct') then $O_g = \mathrm{Ind}_l^g(O'_l)$.

Pf: Lem. In g, fix neutral element H, then the possible X which forms sl_2-triple $\{H, X, Y\}$ forms Zariski-open set in g_2.

Pf. Similar to Mal'cev theorem (any two such triples are conjugate) and prev. thm.,

Idea: Based on preceding discussion, if now $H_l \in h$ is a neutral element realising D' in l, then suffice to ~~show~~ find an $X \in O_g$

$$X = y + z, \quad y \in O'_l, \quad z \in n.$$



$g_2 = l_2 \oplus g_2 \cap n$.

Now X ranges over Zariski-open & projection $\pi : g_2 \to l_2$ is open map (alg geom fact) so set of possible y is open in l_2

But by Lem on l now, set of possible $X_l \in O'_l$ also open in l_2.

Finally l_2 irred. so $\exists\, y \in O'_l$ as desired. □

Defn: O_x even $\Leftrightarrow g_1 = 0 \Leftrightarrow g_{2k+1} = 0 \Leftrightarrow$ WDD has only 0 & 2.
($\alpha(H) = 0$ or 2 for simple) so e.values of ad H-action ...

Lem: Even orbit is Richardson orbit

Pf: Immediate from above.

Prop: Distinguished $\Rightarrow$ even.

Pf: Next pg. (Pf. uses Richardson orbit of Jacobson-Morozov parabolic)

Cor. Distinguished $\Rightarrow$ Richardson (Induced from the g_0 of its Jacobson-Morozov parabolic.)

The inverse map in Bala-Carter can be given by $(l, p_l) \longmapsto G_{ad} \cdot \mathrm{Ind}_{p_l}^{l}(0)$.

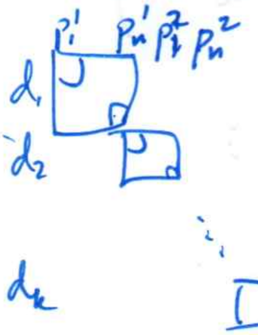
↳ only possible choice as l is just a minimal Levi containing X (which are all G_{ad}-conjugate)

$\mathfrak{sl}_n$ computat^n:

Given Levi $\mathfrak{l}$ (which as mentioned, wrote $\mathfrak{l}(\underline{d})$),
it is hard to compute induced orbit of arbitrary orbit in $\mathfrak{l}$ to $\mathfrak{sl}_n$. (We have done it for Richardson orbit.)
But now w/ transitivity & understanding structure of $\mathfrak{l}$ (as $\times$ of $\mathfrak{sl}_{d_i}$): ..

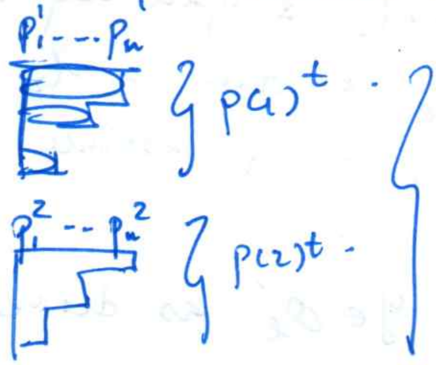
Prop. $\mathcal{O}_\mathfrak{l}$ orbit in $\mathfrak{l}(\underline{d})$, with partitions $p(i) = [p_1^i, \dots, p_n^i]$ corresponding to the $\mathfrak{sl}_{d_i}$ part of $\mathfrak{l}(\underline{d})$.
Then induced orbit $\mathrm{Ind}_{\mathfrak{l}(\underline{d})}^{\mathfrak{sl}_n}(\mathcal{O}_\mathfrak{l})$ has partit^n $\sum p(i)$
$= [\sum_i p_1^i, \dots, \sum_i p_n^i]$.

Pf: By prev. $\mathfrak{sl}_n$ result (applied to $\mathfrak{sl}_{d_i}$), ~~each~~ $\mathcal{O}_\mathfrak{l}$ in $\mathfrak{sl}_{d_i}$ is induced fr. 0 by $\mathfrak{l}(p(i)^t)$



i.e. $\mathcal{O}_\mathfrak{l}$ is induced fr. 0 by $\mathfrak{l}(\overbrace{p(1)^t, \dots, p(k)^t}^{\text{concat.}})$ in $\mathfrak{sl}_n$.

□ By transitivity of induction, we just want the Richardson orbit of $\mathfrak{l}(p(1)^t, \dots, p(k)^t)$ in $\mathfrak{sl}_n$, which has partit^n $(p(1)^t, \dots, p(k)^t)^t$.



$\xrightarrow{t}$ $p_1^1 + p_1^2 + \dots$ ⋮ $p_n^1 + p_n^2 + \dots$ $= \sum p(i)$. □

GOTO pg 9

(Rmk: To any $\mathcal{O}_x$ have well-defined conj. class of parabolic subalg.
Bala-Carter says converse true for distinguished.)