classifying space for phases of matrix product states (MPS).

arXiv: 2501.14241.

Outline. §1. Review of Berry curvature.

§2. Parametrised quantum systems.

§3. 0d case.

§4. Quick intro to tensor networks ~~& MPS~~.

§5. MPS. §6. Properties of MPS.

§1. Berry curvature.

Ex. spin-$\frac{1}{2}$ in a magnetic field.

Hilb. sp.: $\mathbb{C}^2$ qubit

Hamiltonian $H = w_1\sigma^1 + w_2\sigma^2 + w_3\sigma^3 =: \vec{w}\cdot\vec{\sigma}$.

where $\vec{w} = (w_1, w_2, w_3) \in S^2$ represents the magnetic field

$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices.

Eigenvalues $E_\pm = \pm 1$ $\forall\, w \in S^2$. (gapped).

Recall. Bloch sphere $=$ {lines in $\mathbb{C}^2$} $= \mathbb{CP}^1 \cong S^2$ $=$ {(pure) states of qubit}

& $\vec{w}\cdot\vec{\sigma}$ are infinitesimal generators of rotat^n of Bloch sphere about axis $\vec{w}$ ($i\vec{\sigma}$ form basis for $su(2)$)

It follows that $\pm\vec{w}$, viewed as elt. of Bloch sphere, is ~~eigenvalue~~ eigenspace of $H = \vec{w}\cdot\vec{\sigma}$.

Therefore, ~~if~~ we can assemble the gd state spaces of H to form a complex line bundle $\mathcal{L}$ over S^2,

~~essentially~~ ~~we just get~~ the tautological line bundle

and ~~this is~~ $\mathcal{L} \cong$ over $S^2 \cong \mathbb{CP}^1$, !

which is nontrivial

If we vary the magnetic field $\vec{w}$ slowly along a closed path in S^2, adiabatic thm guarantees the system remains in gd state,

but it may pick up a phase : Berry phase.

The evolut^n is governed by (time-dependent) Schrodinger equat^n.

defines a connect^n on $\mathcal{L}$: Berry connect^n

Curvature of this connect^n : Berry curvature Ω.

Recall. Complex line bundles over $X = S^2$ are classified by 1^{st} Chern class

$$= \cancel{\left[\frac{i}{2\pi}\int_X \Omega\right]} \quad \left[\frac{i}{2\pi}\Omega\right]$$

~~curvature~~

curvature 2-form.

$$\in H^2(X, \mathbb{Z}).$$

§2. Parametrised quantum systems

Above is an example of a quantum system **parametrised** by $X = S^2$. X is called the "parameter space".

Think of this ~~$H(\vec{w}) = \vec{w}\cdot\vec{\sigma}$~~ as a map

$$X = S^2 \longrightarrow \{\text{(ground states of) gapped 0d Hamiltonians}\} =: \mathcal{H}_0 .$$

why? → If $X = pt$, then a map

$$X = pt \to \mathcal{H}_0$$

is just a usual quantum system, and homotopy classes of maps $X = pt \to \mathcal{H}_0$ are just usual **phases** of quantum systems.

~~Similarly, define~~
→ When $X = S^1$, can be thought of as adiabatic pumping / Floquet evolution.

Def. A **phase** of a parametrised (0d) quantum system over X is a homotopy class of maps $X \to \mathcal{H}_0$.

Similarly for $\mathcal{H}_d$.

Rmk. For simplicity, restrict to bosonic systems, and do not impose any symmetry.

To understand this, mathematically there are just two goals:

1) Define $\mathcal{H}_d$.

2) Understand the topology of $\mathcal{H}_d$ itself.

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§3. 0d case

~~An~~ 0d system: finite-dim Hilb. sp.

It is reasonable to identify

$$\mathcal{H}_0 = \{\text{0d ground states}\} = \mathbb{CP}^\infty.$$

Now: $\mathbb{CP}^\infty$ has tautological / canonical line bundle $\mathcal{L}_{univ}$, which is universal in the sense that:

$$[X, \mathbb{CP}^\infty] = \left\{\begin{matrix}\text{homotopy classes of} \\ \text{maps } X \xrightarrow{f} \mathbb{CP}^\infty\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{cpx line bundles} \\ \text{over } X\end{matrix}\right\} / \text{iso.}$$

$$f \longmapsto f^* \mathcal{L}_{univ}.$$

Furthermore, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$, so

$$[X, \mathbb{CP}^\infty] \cong H^2(X, \mathbb{Z})$$

1st ~~Chern~~ Chern class!

Upshot: ~~For 0d,~~ the Berry curvature is a complete (topological) invariant for phases of parametrised $d=0$ quantum systems!

What about 1d? Kapustin-Spodyneiko '20: higher Berry curvature, a 3-form on X.
But is this a complete invariant? We don't know until we understand $\mathcal{H}_1$:
Even naively, there is an immediate problem:

Hilb. sp. is infinite-dim'l.

Fundamental problem of many-body quant. ph.: exp growth of Hilb. sp.
To have hope of furnishing a def^n of $\mathcal{H}_1$,
we need a good way to represent gd state of
1d systems "efficient".

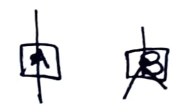
§4. Tensor networks

Tensor network: diagrammatic rep^n of tensors.

rank-r tensor R

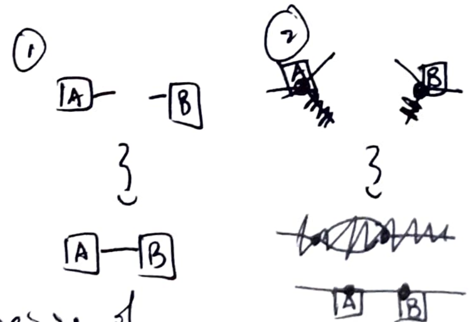
$(r=4)$ vtx with r legs.

tensor product $A\otimes B$



disjoint union.

contraction = tensor product, then take trace



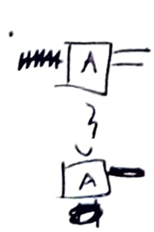
joining two legs.

$\sum A^i_k B^k_i$ = (pairing of vector) under std inner product.

$=\sum A^i_j B^j_k$ (matrix multiplication).

Rmk. No/little distinction between upper & lower indices!

Grouping of legs: E.g.



$d_1\times d_2$ matrix i.e. rk-2 tensor can be viewed as vector in $\mathbb{C}^{d_1\times d_2}$ i.e. rk-1 tensor

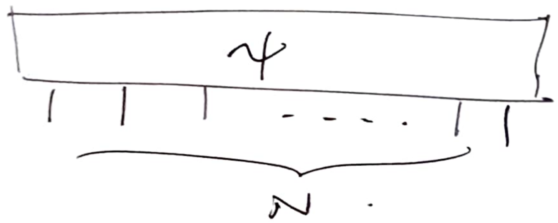
§5. Matrix product states (MPS)

1d lattice system

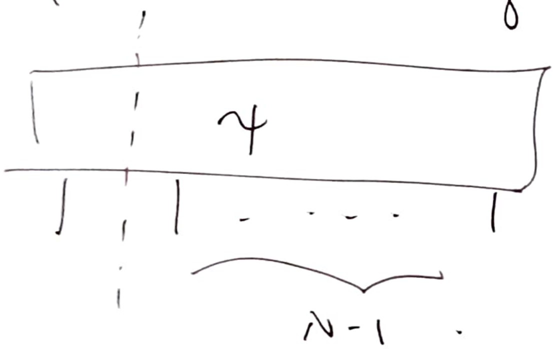
~~Ex. Singular value decomposit' (SVD)~~ .

Suppose $|\psi\rangle$ is a state of 1d lattice system on N lattice sites.

Represent as rank-N tensor (using std basis for $\mathbb{C}^{d_1} \dots \mathbb{C}^{d_N}$)

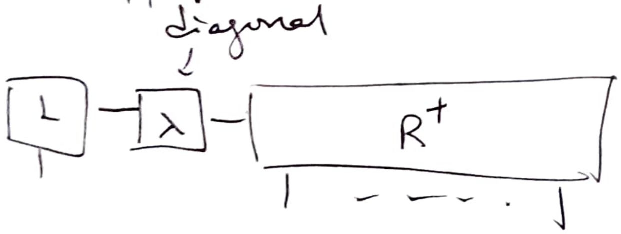


Group last (N-1) legs:



So now view ψ as a rk-2 tensor = matrix.

Now apply singular value decomposit^{n} (SVD) to this matrix.

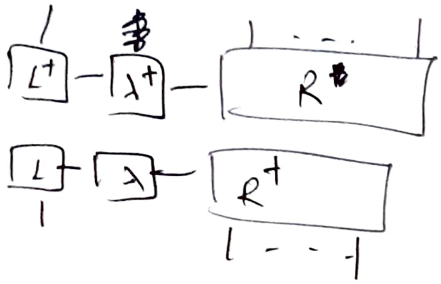


This is also known as Schmidt decomposit^{n}

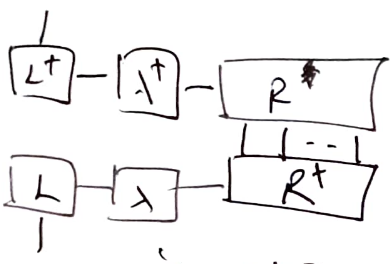
$$|\psi\rangle = \sum_i \lambda_i |L_i\rangle \otimes |R_i\rangle$$

for $\{|L_i\rangle\}$, $\{|R_i\rangle\}$ orthonormal.

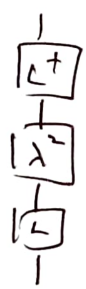
→ Note density matrix of $|\psi\rangle$ is

$\rho =$ 

Reduced density matrix ρ_L : take trace over R:

$\rho_L =$ 

Recall R ~~unitary~~ isometric (by SVD), so

$\rho_L =$ 

$= \sum \lambda_i^2 |L_i\rangle\langle L_i|$. , $\sum \lambda_i^2 = 1$.

(von Neumann) entanglement entropy

$$= -\mathrm{tr}(\rho_L \log \rho_L) = -\sum \lambda_i^2 \log(\lambda_i^2) .$$

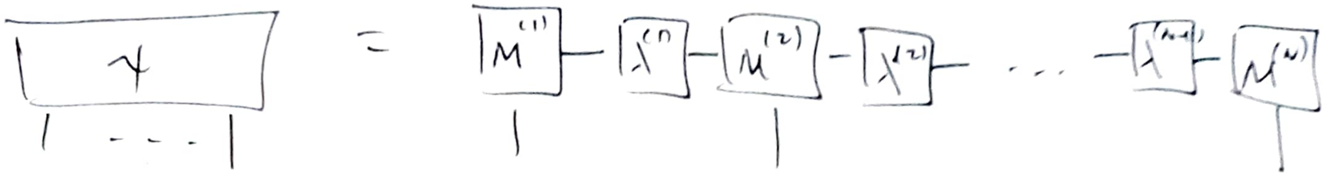
entanglement rank = Schmidt no.

= # non-zero λ_i .

quantifies the amount of entanglement along the cut of the 1d system.

Power of tensor networks:

- can do this for any cut
- can do this for all cuts!



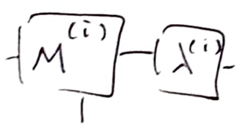
$\lambda^{(i)}$ quantifies entanglement across cut between i and $(i+1)^{th}$ lattice site.

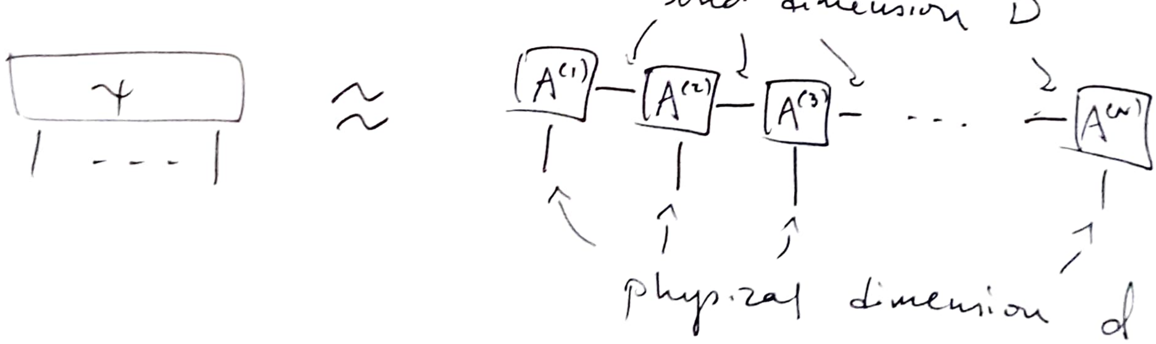
Hastings, '07: gd state of gapped 1d Hamiltonian satisfies area law for entanglement entropy.

i.e. bounded across any cut.

So truncate the $\lambda^{(i)}$ to have fixed/const. rk D, called bond dimension

This will give good approximat^n to gd states of gapped 1d Hamiltonian.

Can contract  to form new tensor 



This is a matrix product state (MPS).

§6. Properties of MPS

The $A^{(i)}$ are rk 3 tensors, of dimension $D \times D \times d$.

~~Simplification 1: Restrict to translationally-invariant states/systems.~~

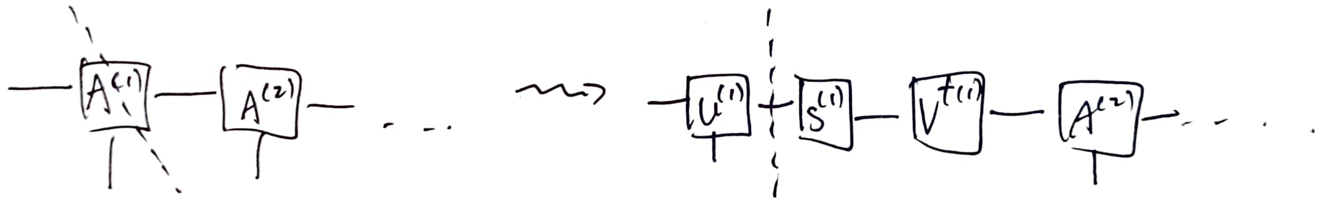
~~i.e. all $A^{(i)} = A$.~~

Write $A^{(i);j}_{\alpha\beta}$ for $1 \le \alpha, \beta \le D$, $1 \le j \le d$.

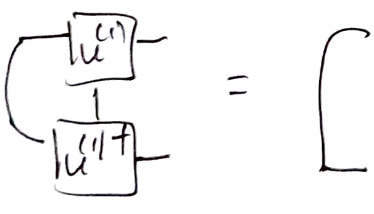
Still many deg. of freedom, and not particularly efficient rep^{n}.

Would like to get a canonical form for MPS.

Use SVD.



$u^{(1)}$ isometric means



or $\sum u^{j\dagger} u^{j} = \mathbb{I}_{D\times D}$ as $D \times D$ matrices.

In other words, we can enforce

$\sum A^{j\dagger} A^{j} = \mathbb{I}_{D\times D}$

for all MPS tensor A.

This is called left-canonical/-normalised/-isometric form.

Similarly, can also enforce right-canonical form:

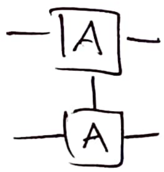
$$\sum A^j A^{j\dagger} = \mathbb{1}_{D\times D}.$$

Simplification: Assume translation-invariance,
i.e. all $A^{(i)} = A$.

Why is left-normalisation useful/impt?

Two-pt correlator $\langle\psi| O_0 O_{j+1} |\psi\rangle$. for operator O.

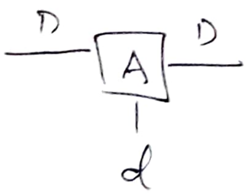
$=$ 

Transfer matrix $E :=$ 

(as in stat. mech.)

Left-norm. means E has left eigenvector $\mathbb{1}$ w/ eigenvalue 1

If A injective as a map $\mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$,



$\Leftrightarrow A^1, \ldots, A^d$ span the v.s. of $D\times D$ matrices,
(this is a generic/open condition on A),

then it turns out E has largest e-value $\lambda_1 = 1$ w unique e.vector
and second-largest e.val. $|\lambda_2| < 1$. (
(Perron-Frobenius style argument)
[Rmk: close to an iff, in a technical sense which I won't go into.]

Then as $j \to \infty$, corr. f! is controlled by $|\lambda_2|$.
Corr. length $\xi = -\dfrac{1}{\log|\lambda_2|}$.
i.e. the state has exponentially decaying correlat!s,
as expected of (unique) gd state of gapped Hamiltonian (Hastings).

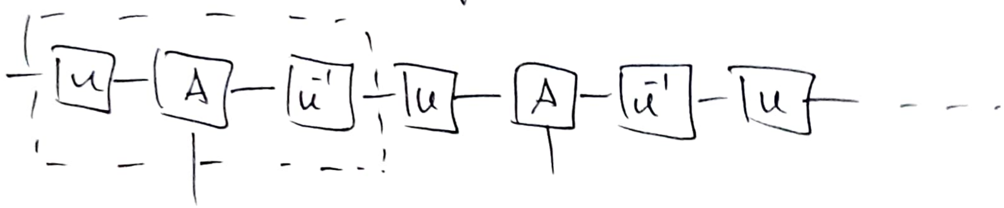
Upshot: Consider MPS which are:

1. left/right-normalised (canonical form, & makes analysis of transfer matrix easier)
2. translat!-invariant (reasonable assumpt! on the system)
3. injective (generic/open condit!)

These corr. to (unique) gd states of gapped 1d Hamiltonian.

Gauge freedom

We can do change of basis on the bond dimensions:



Tensor A and UAU^{-1} give the same state $|\psi\rangle$.

Gauge transformat$^{\circ}$: $A^j \longmapsto \lambda U A^j U^{-1}$.

where U unitary to preserve left/right normalisat$^{\circ}$

& $\lambda \in U(1)$ just an overall phase factor.

Next: Build moduli sp. $\mathcal{H}_1$ of these MPS tensors / gauge transformat$^{\circ}$s.