~~Define regular semisimple~~

Preliminaries

① Adjoint group / inner automorphisms.

$$\begin{array}{ccc} G & \xrightarrow{Ad} & Aut(\mathfrak{g}) \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow[ad]{} & End(\mathfrak{g}) \end{array}$$

G_{ad} = connected Lie group assoc to $ad(\mathfrak{g})$

← local diffeomorphism at 0.

generated by open neighbourhood of identity

G_{ad} is generated by $\exp(ad\, X)$ for $X \in \mathfrak{g}$.

Group of inner automorphisms $Int(\mathfrak{g})$
= generated by $\exp(ad\, X)$ for $ad\, X$ nilpotent (arbitrary field k) of char 0.

next pg for covering group first

③ Zariski topology.
$\mathfrak{g}$ is finite-dim over k of dim n
↳ Identify w/ affine n-space $\mathbb{A}^n$ and equip w/ Zariski topology.
Key pt: non-empty open sets are dense!

④ Semisimple elements / Cartan subalgebra.
Any X semisimple lies in a Cartan subalgebra.
Lemma: $\mathfrak{g}$ reductive $\Rightarrow$ $\mathfrak{g}^X$ reductive (Humphreys exercise 8.7).
Idea: X lies in CSA $\mathfrak{h}$, roots Φ. Recall root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_\alpha$.
For $W \in \mathfrak{g}_\alpha \cap \mathfrak{g}^X$,
$$0 = [X, W] = \alpha(X) W$$
Deduce:
$$\mathfrak{g}^X = \mathfrak{h} \oplus \sum_{\alpha(X)=0} \mathfrak{g}_\alpha$$
← root system.

Center of $\mathfrak{g}^X = \mathfrak{h}_1 = \bigcap_{\alpha(X)=0} \ker\alpha \subset \mathfrak{h}$.

$$\mathfrak{h}_2 = \sum_{\substack{\alpha(X)=0 \\ \alpha \text{ simple}}} \mathbb{C}[H_\alpha] \qquad \mathfrak{g}^X = \mathfrak{h}_1 \oplus \underbrace{\mathfrak{h}_2 \oplus \sum_{\alpha(X)=0} \mathfrak{g}_\alpha}_{\text{semisimple}}$$

Next pg for example.

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② Covering group:

$\exists$ simply connected Lie group $G_{sc} = \tilde{G}$ with Lie algebra $\mathfrak{g}$

covering group of G_{ad}.

"largest" Lie group with Lie algebra $\mathfrak{g}$

(G_{ad} is "smallest")

~~any~~ simply connected $G \xrightarrow{\Pi} H$

$\mathfrak{g} \xrightarrow[\pi]{} \mathfrak{h}$

can lift or ~~integrate~~ 'integrate' to $\Pi: G \to H$

Idea: look at $G \times H$

Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$

and graph $\mathfrak{j}$ of π $((X, \pi(X))$

is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$

$\Rightarrow \exists$ Lie subgroup $J \subset G \times H$ corresponding

Then $\pi_1: J \to G$ is isomorphism (use G simply connected)

so $\pi_2: J \to H$ is the desired Π.

(Only use this once later!)

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④ Example 2.2: $\mathfrak{g} = \mathfrak{sl}_4$.

$X_1 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 2 & \\ & & & -2 \end{pmatrix}$ No root α kills it $\because$ e.values all distinct

So $\mathfrak{g}^{X_1}$ is Cartan $\equiv \mathfrak{h}$ (dim 3) $\longrightarrow$ later will see this again in more general context.

$X_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 0 \end{pmatrix}$ Only $\pm(\varepsilon_1 - \varepsilon_2)$ kills it.

$\mathfrak{g}^{X_2} = \mathfrak{h} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_2 - \varepsilon_1}$.

Can see this as block matrix

$\begin{pmatrix} * & * & 0 \\ * & * & \\ 0 & & * & \\ & & & * \end{pmatrix}$ dim 5.

Center of $\mathfrak{g}^{X_2} = \mathfrak{h}_1 = \bigcap \ker \alpha = \begin{pmatrix} a & & 0 \\ & a & \\ 0 & & * \\ & & & * \end{pmatrix}$

$\mathfrak{h}_2 = \mathbb{C}\left[\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\right]$

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Classifying semisimple orbits

Motivating example : sl_n.

Clearly, conjugacy classes of semisimple elements (diagonalisable)

just diagonal matrices modulo permutation (S_n) — $\mathfrak{h}$. W .

$\varepsilon_i \in \mathfrak{h}^*$

$\varepsilon_i : e_{jj} \longmapsto \delta_{ij}$

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ simple roots.

~~How does reflection~~

$\sigma_{\alpha_i} \in W$

act on $\mathfrak{h}$?

(Identify $\mathfrak{h} \cong \mathfrak{h}^*$ via Killing form)

$$\sigma_{\alpha_i} : \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix} \longmapsto -\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix}$$

$$H_{\alpha_i} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix}$$

elements in $\mathfrak{h}$ correspond to those which are orthogonal to α_i in $\mathfrak{h}^*$?

Answer: $\ker \alpha_i$ (because up to scalar multiple, $\kappa(H_{\alpha_i}, h) = \alpha_i(h)$.)

So σ_{α_i} fixes $\begin{pmatrix} \ddots & & \\ & x & \\ & & x \\ & & & \ddots \end{pmatrix}$.

Note $\ker \alpha_i \oplus \mathbb{C}[H_{\alpha_i}] = \mathfrak{h}$.

Effect of σ_{α_i}:

$$\begin{pmatrix} \ddots & & \\ & x+y & \\ & & x-y \\ & & & \ddots \end{pmatrix} \longmapsto \begin{pmatrix} \ddots & & \\ & x-y & \\ & & x+y \\ & & & \ddots \end{pmatrix}$$

i.e. it transposes $(i \;\; i+1)$!

So W acts exactly on $\mathfrak{h}$ like our desired semisimple orbits.

[of any semisimple element lies in a $\mathfrak{h}$.]

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Therefore:

Theorem: $\mathfrak{h}/W \longleftrightarrow \{\text{semisimple orbits}\}$

$[X] \longmapsto \mathcal{O}_X$

is a bijection.

Proof: Three things to check:

① Well-defined.

Just need: if W-conjugate, then G_{ad}-conjugate.

② Surjective.

Given X semisimple, (it lies in a CSA $\mathfrak{h}'$).

Need: X', ~~a expect $\mathfrak{h}'$ to be~~

to be G_{ad}-conjugate to X

Expect $\mathfrak{h}'$ to be G_{ad}-conjugate to $\mathfrak{h}$.

→ Chevalley's theorem.

③ Injective.

Suppose $\mu([X_1]) = \mu([X_2])$.

$\mathcal{O}_{X_1} = \mathcal{O}_{X_2}$.

① $x \cdot X_1 = X_2$ ($x \in G_{ad}$).

③ $X_2 \in \mathfrak{h}, x\cdot\mathfrak{h} \subset \mathfrak{g}^{X_2}$ ($\because \mathfrak{h}, x\cdot\mathfrak{h}$ CSA hence abelian)

③ This is the crux of the problem, that x does not fix $\mathfrak{h}$ and $x\cdot\mathfrak{h} \neq \mathfrak{h}$. Want to make $\mathfrak{h}, x\cdot\mathfrak{h}$ further conjugate.

Philosophically: if G_{ad}-conjugate, then W-conjugate?

W only acts on $\mathfrak{h}$.

② Now want to show: are X_1, X_2 W-conjugate?

No reason to expect this if X_1, X_2 a priori only G_{ad}-conjugate.

As $X_1, X_2 \in \mathfrak{h}$, need them to be conjugate via sth that fixes $\mathfrak{h}$.

④ $\mathfrak{h}, x\cdot\mathfrak{h}$ conjugate via adjoint group of $\mathfrak{g}^{X_2}$, say via y.

$y\cdot x\cdot\mathfrak{h} = \mathfrak{h}$

But y is inner automorphism of $\mathfrak{g}^{X_2}$

so fixes X_2

$y\cdot x\cdot X_1 = y\cdot X_2 = X_2$

$\therefore (y\cdot x)$ fixes $\mathfrak{h}$ and X_1, X_2 conjugate via $(y\cdot x)$.

Just need: if G_{ad}-conjugate & fix $\mathfrak{h}$, then W-conjugate.

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Key tools.

Lemma ①: If W-conjugate, then G_{ad}-conjugate.

Pf: Have seen this before, one way or another: suffice show for reflect^n σ_α (α root.)

Inner automorphism

$$\tau_\alpha = \exp(\mathrm{ad}\, x_\alpha)\exp(\mathrm{ad}\,(-y_\alpha))\exp(\mathrm{ad}\, x_\alpha)$$

~~Easy~~ $\tau_\alpha(h_\alpha) = -h_\alpha$ ($\mathfrak{sl}_2$ computat^n).

~~Recall~~

Recall those orthogonal to h_α are $\ker\alpha$ ($\mathfrak{h} = \ker\alpha \oplus \mathbb{C}[h_\alpha]$)

Clearly $\tau_\alpha(h) = h$ for $h \in \ker\alpha$. ✓

Lemma ②: Chevalley's theorem.

Various proofs:

a) Humphreys c.16 'elementary' but very long!

Chevalley ⟵ b) Analytic (in ref. of text.) — only for $\mathbb{C}$ } similar idea.

c) Algebraic geometry

Jacobson chapter IX ⟵ arbitrary char $k = 0$, $k = \bar{k}$

— use regular elements, Zariski topology.

Chevalley's theorem

Regular elements

For $X \in \mathfrak{g}$, consider char. poly of ad X (in t, say)

Degree $= \dim_k \mathfrak{g}$, const. term $= \det(\mathrm{ad}\, X) = 0$ ($\because [XX] = 0$)

$$\chi(\mathrm{ad}\, X) = t^d + c_{d-1}(X)\, t^{d-1} + \dots + c_1(X)\, t.$$

— Coefficients c_i are homogeneous polynomials on $\mathfrak{g}$ of deg $d-i$.

Pf: Fix basis $X_1, \dots, X_d$ of $\mathfrak{g}$,

$$X = \sum a_i X_i$$

ad $X(X_j)$ is linear combinat^n of a_i

so matrix for ad X has entries linear combinat^n of a_i

- For $\phi \in \mathrm{Aut}(\mathfrak{g})$,

$$\mathrm{ad}(\phi(x)) = \phi \circ \mathrm{ad}\, x \circ \phi^{-1}$$

so $\chi(\mathrm{ad}(\phi(x))) = \chi(\mathrm{ad}\, x)$, i.e. the c_i are $\mathrm{Aut}(\mathfrak{g})$-invariant.

Def^n: - Min r s.t. $c_r \neq 0$ is <u>rank</u> of $\mathfrak{g}$.

Def^n: $X \in \mathfrak{g}$ is <u>regular</u> (in textbook, regular semisimple) iff $c_r(X) \neq 0$.

Set of regular elements is Zariski-open and G_{ad}-invariant.

Illustrat^n: - Generalised eigenspace decomposit^n w.r.t. ad X:

$$\mathfrak{g} = \mathfrak{g}_0^X \oplus \bigoplus_{\lambda \in k} \mathfrak{g}_\lambda^X$$

Regular $\iff$ $\dim \mathfrak{g}_0^X = rk(\mathfrak{g})$ (else $\geq$).

textbook defines regular as $\dim \mathfrak{g}_0^X = rk\,\mathfrak{g}$. Can show easily that regular + semisim. = regular semisim.

If X regular, $c_r(X) = \prod \lambda$ (I haven't been careful w/ signs for simplicity)

In particular if $X \in \mathfrak{h}$; then $c_r(X) = \prod_{\alpha \text{ non zero root}} \alpha(X)$ (2.1.10)

Example: $\mathfrak{sl}_n$.

$\chi(\mathrm{ad}\, X) = \chi(\mathrm{ad}\, X_s)$

say X_s has eigenvalues $\lambda_1, \ldots, \lambda_n$ (i.e. these are the generalised eigenvalues of X)

Then $\mathrm{ad}\, X_s(E_{ij}) = (\lambda_i - \lambda_j) E_{ij}$

$\mathrm{ad}\, X_s(E_{ii} - E_{i+1,i+1}) = 0$.

So $\mathrm{ad}\, X_s$ has eigenvalues

$$\underbrace{0, \ldots, 0}_{\substack{n-1 \\ = \\ rk(\mathfrak{sl}_n) \\ = \\ (\dim \text{ CSA of } \mathfrak{sl}_n)}}, \quad \lambda_i - \lambda_j$$

$c_r(X) = \prod(\lambda_i - \lambda_j)$, regular iff e.values distinct

rk. also it is hence semisimple!

E.g. $\mathfrak{sl}_2$, basis $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $X = \begin{pmatrix} & 1 \\ & \end{pmatrix}$, $Y = \begin{pmatrix} & \\ 1 & \end{pmatrix}$

$Z = \begin{pmatrix} t & c \\ d & -t \end{pmatrix}$. $\chi(Z) = \lambda^2 - (t^2 + cd)$

$\lambda_1 = +\sqrt{t^2+cd}$, $\lambda_2 = -\sqrt{t^2+cd}$, $\prod(\lambda_i - \lambda_j) = -4(t^2+cd)$

cf. 2.1.14

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4 Generation of CSAs

Note sl_n example: $X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ regular iff λ_i distinct.

Then $g^X = \{$diagonal matrices$\} = h$

Else if some $\lambda_i = \lambda_j$, g^X is larger!.

Lemma 2.1.9 (generation of Cartan subalgebra). X regular

$g = \bigoplus_{\lambda \in k} g^X_\lambda$ generalised e-space decomposit^n of g w.r.t. ad X.

$= h \oplus V$

Claim: $h = g_0^X$ is CSA.

Check: $[g^X_\mu, g^X_\lambda] \subset g^X_{\mu+\lambda}$

So ad h acts blockwise

$[h, g^X_\lambda] \subset g^X_\lambda$.

Pf. Consider:

$A = \{h \in h \mid \text{ad } h|_V \text{ non-singular}\}$ (Zariski-open)

$\ni X$

$B = \{h \in h \mid \text{ad } h|_h \text{ not nilpotent}\}$ (Zariski-open (equiv to $\chi(\text{ad } h) \neq t^k$))

If h not nilpotent, $\exists b \in B$ (Engel's thm.)

So $A \cap B \neq \emptyset$

$\ni c$

ad $c|_V$ non-singular, ad $c|_h$ not nilpotent

$\Downarrow$

$g_0^c \subsetneq h$ contradict regularity of X (has minimal zero eigenspace dimension!)

So h nilpotent

Now show $N_g(h) = h$.

If $y \in N_g(h)$ then $[X, Y] \in h$ ~~$[Y, X]$~~

But ad $X|_h$ nilpotent

want to show $y \in g_0^X$, or $(\text{ad } X)^M(Y) = 0$ $\exists M$

So $\exists N$ with

$0 = (\text{ad } X)^N([X, Y])$

$= (\text{ad } X)^{N+1}(Y)$

$\Rightarrow Y \in g_0^X = h$ as desired.

Impt remark: If ~~g~~ g semisimple, then ad X semisimple $\Rightarrow X$ semisimple (being in CSA g^X) & $h = g^X$

Hence textbook calls them 'regular semisimple'

Easy corollary ('uniqueness' of general^n of CSA): $\mathfrak{h}$ CSA, if $X \in \mathfrak{h}$ regular, then $\mathfrak{h} = \mathfrak{g}_0^X$.

Pf: $\mathfrak{h}$ nilpotent so $\mathrm{ad}X|_{\mathfrak{h}}$ nilpotent so $\mathfrak{h} \subseteq \mathfrak{g}_0^X$, but $\mathfrak{g}_0^X$ is CSA so $\mathfrak{h} = \mathfrak{g}_0^X$ ($\mathfrak{h}$ is maximal nilpotent) □

Theorem (Chevalley, conjugacy of CSA): Any two CSAs $\mathfrak{h}_1, \mathfrak{h}_2$ are conjugate by inner automorphisms / G_{ad}

i.e. $\exp(\mathrm{ad}\, X)$ for $\mathrm{ad}\, X$ nilpotent

Pf: Lemma (algebraic geom / inverse f^n thm.)

Let $f: \mathbb{A}^n \to \mathbb{A}^n$ be polynomial map / regular map
$\;\;\;\;\;\;\;\;\; = (f_1, \dots, f_n)$

If Jacobian $df = \left(\frac{\partial f_i}{\partial x_j}\right)$ invertible at some pt a, then $\mathrm{im}(f)$ contains Zariski-open set

Sketch: ① $f_1, \dots, f_n$ algebraically indpt. by invertibility of Jacobian

(Else, choose F of min deg s.t. $F(f_1, \dots, f_n) = 0$ then $\left(\frac{\partial F}{\partial y_i}(f_1, \dots, f_n)\right)$ always in kernel of Jacobian, so $\frac{\partial F}{\partial y_i}(f_1, \dots, f_n) \equiv 0$

$\Rightarrow \frac{\partial F}{\partial y_i} = 0 \Rightarrow F = 0$)

② $k(x_1, \dots, x_n) / k(f_1, \dots, f_n)$ have same tr. deg. so algebraic & finitely-generated so finite

③ So $\exists g$ s.t.

$k[f_1, \dots, f_n, g^{-1}] \subset k[x_1, \dots, x_n, g^{-1}]$ integral

i.e. corresponds to finite morphism into ~~onto~~ $\{g \neq 0\}$

But finite morphism is surjective!

So $\mathrm{Im}(f)$ contains $\{g \neq 0\}$. □

Now fix any CSA $\mathfrak{h}$.

Key step: Consider conjugat^n map:

$$f: \mathfrak{g} \to \mathfrak{g}$$

$$(x_i \in k) \qquad \underbrace{h}_{\mathfrak{h}} + \sum x_j \underbrace{b_j}_{\text{Basis for each root space } \mathfrak{g}_\alpha^{\mathfrak{h}}} \longmapsto e^{x_1 \mathrm{ad} b_1} \cdots e^{x_n \mathrm{ad} b_n}(h)$$

$\alpha \in \mathfrak{h}^*$ root

(if $\mathfrak{h}$ only nilpotent, still have generalised root space decomposit^n)

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5 **Remark:** In analytic proof, also consider conjugat^n map

$$\varphi : G \times \mathfrak{h} \to \mathfrak{g}$$
$$(x, H) \mapsto x \cdot H$$

& consider its differential $d\varphi$!.

Now: consider Jacobian / differential of f at a pt a

In direct^n $h + b$, this is
$$\underset{\parallel}{} \\ \textstyle\sum x_j b_j$$

$$\left.\frac{d}{dt} f(a + t(h+b))\right|_{t=0}$$

~~expand to~~ can ignore all t^2 and above terms

$$\prod (I + t x_j \,\mathrm{ad}\, b_j)(a + th)$$

So derivative is coefficient of linear term t

$$= h + \textstyle\sum x_j \,\mathrm{ad}\, b_j (a)$$
$$= h + [b, a] \quad (= h - [a, b])$$

Remark: Computat^n is similar in the analytic case!

$\Rightarrow$ differential of f, df ~~is it~~ acts as id on $\mathfrak{h}$ and $-\mathrm{ad}\, a$ on root spaces $\mathfrak{g}_\alpha$

eigenvalues $\alpha(a)$.

So suffices $\alpha(a) \neq 0$ $\forall$ roots α.

But only finitely many roots α_i & each $\alpha_i(a) \neq 0$ is a Zariski-open condition! $\square$

To conclude: Chevalley's thm: for any two CSAs $\mathfrak{h}_1, \mathfrak{h}_2$ w/ corresponding conjugat^n maps f_1, f_2, their image contains Zariski-open, and regular elements are Zariski-open, so $\exists$ regular x, $x_1, x_2 \in G_{ad}$, $h_1, h_2 \in \mathfrak{h}_1, \mathfrak{h}_2$, s.t.

* unique CSA $\mathfrak{g}_0^x$! $\Longleftarrow$ $x = x_1 \cdot h_1 = x_2 \cdot h_2$

$\mathfrak{h}_1, \mathfrak{h}_2$ conjugate via ~~x~~ x_1, x_2 to $\mathfrak{g}_0^x$, hence to each other.

cont. on last pg 7

$N_{G_{ad}}(\mathfrak{h})$

Back to :

Lemma 3 : If G_{ad}-conjugate & fix $\mathfrak{h}$, then W-conjugate.

Rk: Algebraic proof (for arbitrary k) in Bourbaki Chap 8 Sect 5
→ "elementary" but v. long!

Analytic proof : use covering group & irred. f.d. rep!!

First : suppose automorphism $x \in G_{ad}$ leaves $\mathfrak{h}$ invariant.

x acts on roots $\mu \in \mathfrak{h}^*$ by

$x \cdot \mu = (h \mapsto \mu(x^{-1} h))$

x must send simple roots (base) Δ to another base of simple roots, $x \cdot \Delta$.
positive roots to another system of positive roots.

$x \cdot \mathfrak{g}_\alpha = \mathfrak{g}_{x \cdot \alpha}$.

$v \in \mathfrak{g}_\alpha$
$\mathrm{ad}(H) v = \alpha(H) v$
$x^{-1} \mathrm{ad}(H) \cdot x^{-1} \cdot x v = x \alpha(H) v$
$(\mathrm{ad}\, x^{-1} H) x \cdot v = \alpha(H) x \cdot v$
$\qquad (x \cdot \alpha)(x \cdot H)$

But Weyl group W acts transitively on bases,
$\exists\ w \cdot \Delta = x \cdot \Delta$

And by Lemma 1, $\exists\ y \in G_{ad}$ with same act^n as $w \in W$

So $\underbrace{y^{-1} x}_{t}$ leaves Δ invariant.
Want to show x & y have same act^n on $\mathfrak{h}$, then w is the desired.

So suffice show : if $t \in G_{ad}$ leaves Δ invariant, then $t|_{\mathfrak{h}}$ is identity.

Suffice show : $t \cdot \lambda = \lambda \quad \forall$ dominant integral λ
(choose usual basis for $\mathfrak{h}$)

Key idea : consider irred. f.d. rep of $\mathfrak{g}$ w highest weight λ!!

$$\mathfrak{g} \xrightarrow{\pi_\lambda} \mathrm{End}(V)$$

$\tilde{G}$ s.c. so can 'lift' or 'integrate' to a Lie group map $\tilde{\pi}_\lambda$

and $\quad \tilde{G} = G_{sc} \xrightarrow{\tilde{\pi}_\lambda} \mathrm{Aut}(V)$

$x \in$ simply connected covering group of G_{ad}

$\mathrm{Ad}(x) = t$

6.
$$\begin{tikzcd} \tilde{G} \arrow[r, "\bar{\pi}_\lambda"] \arrow[d, "\Phi_x"'] & \mathrm{Aut}(V) \arrow[d, "\Phi_{\bar{\pi}_\lambda(x)}"] \\ \tilde{G} \arrow[r, "\bar{\pi}_\lambda"'] & \mathrm{Aut}(V) \end{tikzcd}$$

conjugat^{n} by x

$$\begin{tikzcd} X \in \mathfrak{g} \arrow[r, "\pi_\lambda"] \arrow[d, "t = \mathrm{Ad}(x)"'] & \mathrm{End}(V) \arrow[d, "\mathrm{Ad}(\bar{\pi}_\lambda(x))"] \\ \mathfrak{g} \arrow[r, "\pi_\lambda"'] & \mathrm{End}(V) \end{tikzcd}$$

just conjugat^{n} by $\bar{\pi}_\lambda(x)$.

$\pi_\lambda(t\cdot X) = \bar{\pi}_\lambda(x)\, \pi_\lambda(X)\, \bar{\pi}_\lambda(x)^{-1} \Rightarrow \pi_\lambda(t\cdot X)\, \bar{\pi}_\lambda(x) = \bar{\pi}_\lambda(x)\, \pi_\lambda(X)$ $\forall X \in \mathfrak{g}$.

if $v \in V_\mu$ (weight space of V).

then

$\pi_\lambda(t\cdot \cancel{X} H)\, \bar{\pi}_\lambda(x)(v) = \bar{\pi}_\lambda(x)\, \pi_\lambda(\cancel{X} H) v$

$= \mu(H)\, \bar{\pi}_\lambda(x)(v)$.

$\Rightarrow \bar{\pi}_\lambda(x)(v) \in V_{t\cdot\mu}$.

So $\forall \mu$, $V_\mu \simeq V_{t\cdot\mu}$ via $\bar{\pi}_\lambda(x)$! (1).

Consider highest weight/maximal vector $v^+ \in V_\lambda$.
But now t fixes simple, positive roots so fixes the Borel subalgebra $\mathfrak{b}$ (which kills v^+)

$\Rightarrow \bar{\pi}_\lambda(x) v^+$ is still killed by all $\mathfrak{b}$

$\Rightarrow \exists$ is still maximal

$\Rightarrow V_\lambda \simeq V_\lambda$ via $\bar{\pi}_\lambda(x)$! (2).

---

(1) + (2) : $\lambda = t\cdot\lambda$ as desired.

(from pg 5)

7 $\underline{\text{Cor}}$ to Chevalley's thm.

(12-1-12)

Any CSA is $\mathfrak{g}_0^X$ for some X regular :

$\underline{\text{Pf}}$. Any $\mathfrak{h}$ is conjugate to $\mathfrak{g}_0^{X'}$, X' regular, then the image of X' under this conjugat^n is the desired X. $\square$