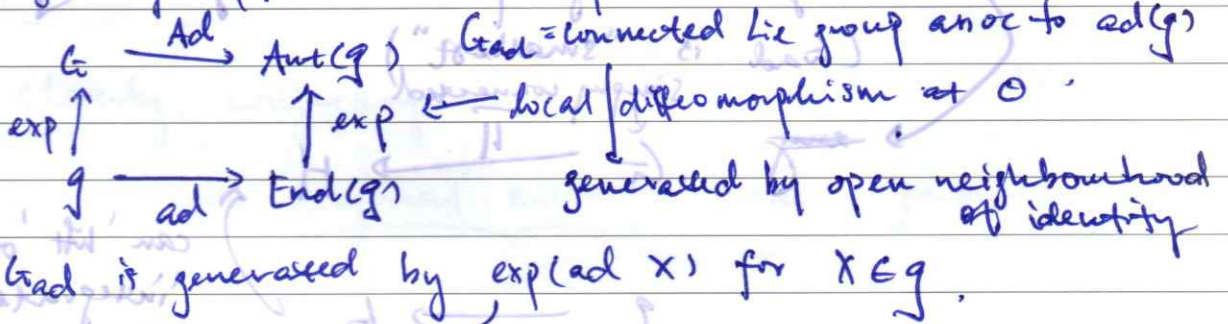


~~Define \mathfrak{g} as a Lie algebra~~

Preliminaries

① Adjoint group / inner automorphisms



Group of inner automorphisms $\text{Int}(\mathfrak{g})$

= generated by $\exp(\text{ad } X)$ for $\text{ad } X$ nilpotent (arbitrary field k of char 0).

next pg for covering group first

② Zariski topology

\mathfrak{g} is finite-dim over k of dim n

\hookrightarrow identify \mathfrak{g} affine n -space \mathbb{A}^n

and equip \mathfrak{g} in Zariski topology

Key pt: non-empty open sets are dense!

④ Semisimple elements / Cartan subalgebra

Any X semisimple lies in a Cartan subalgebra.

Lemma: \mathfrak{g} reductive $\Rightarrow \mathfrak{g}^X$ reductive

(C Humphreys exercise 8.7)

Idea: X lies in CSA \mathfrak{h} , roots Φ . Recall root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

For $W \in \mathfrak{g}_{\alpha} \cap \mathfrak{g}^X$, $0 = [X, W] = \alpha(X)W$

Deduce:

$$\mathfrak{g}^X = \mathfrak{h} \oplus \sum_{\alpha(X)=0} \mathfrak{g}_{\alpha}$$

Center of $\mathfrak{g}^X = \mathfrak{h}_1 = \bigcap_{\alpha(X)=0} \ker \alpha \subset \mathfrak{h}$

$$\mathfrak{h}_2 = \sum_{\substack{\alpha(X)=0 \\ \alpha \text{ simple}}} \mathbb{C}[\alpha]$$

$$\mathfrak{g}^X = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \sum_{\alpha(X)=0} \mathfrak{g}_{\alpha}$$

Next pg for example

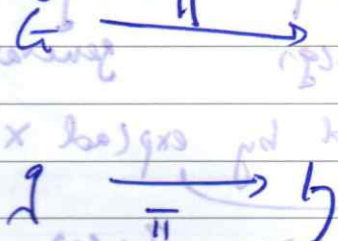
② Covering group

\exists simply connected Lie group $G_{sc} = G$ w
Lie algebra \mathfrak{g}

covering group of G_{sc} .

"largest" Lie group w Lie algebra \mathfrak{g}

(G_{sc} is "smallest")
Simply connected



can 'lift' or 'integrate' to
 $\pi: G \rightarrow H$

Idea: look at $G \times H$

Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$

and graph J of π $(X, \pi(X))$
is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$

$\Rightarrow \exists$ Lie subgroup $J \subset G \times H$
corresponding

Then $\pi_1: J \rightarrow G$ is isomorphism
(use G simply connected)

So $\pi_2: J \rightarrow H$ is the desired π .

(Only use this once later!)

③ Example 2.2: $\mathfrak{g} = \mathfrak{sl}_2$

$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

No α kills it \because e-values all distinct

So \mathfrak{g}^{X_1} is Cartan $\cong \mathfrak{h}$
(dim 3)

later will see this again in more general context

$X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

Only $\pm(\epsilon_1 - \epsilon_2)$ kills it.

$\mathfrak{g}^{X_2} = \mathfrak{h} \oplus \mathfrak{g}_{\epsilon_1 - \epsilon_2} \oplus \mathfrak{g}_{-\epsilon_1 + \epsilon_2}$

Can see this as block matrix

$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix}$

dim 5

Center of $\mathfrak{g}^{X_2} = \mathfrak{h}_1 = \mathfrak{h}$ kernel

$\mathfrak{h}_2 = \mathbb{C} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} a & 0 \\ 0 & * \end{pmatrix}$

2

Classifying semisimple orbits

Motivating example : $sl_n \rightarrow [X]$

clearly, conjugacy classes of semisimple elements (diagonalisable)

just diagonal matrices modulo permutation (S_n)

$\xi_i \in \mathfrak{h}^*$
 $\xi_i : \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \mapsto \delta_{ij}$

$\alpha_i = \xi_i - \xi_{i+1}$ simple roots

How does reflection

act on \mathfrak{h} ? $\sigma_{\alpha_i} \in W$
 (Identify \mathfrak{h} with \mathfrak{h}^* via Killing form)

$\sigma_{\alpha_i} : \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \mapsto - \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$
 $H_{\alpha_i} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$

elements in \mathfrak{h} correspond to those which are orthogonal to α_i in \mathfrak{h}^* ?
 Answer: ker α_i became up to scalar multiple, $\kappa(H_{\alpha_i}, \mathfrak{h}) = \mathfrak{d}(\mathfrak{h})$

So σ_{α_i} fixes $\begin{pmatrix} \dots & \dots & \dots \\ \dots & x & \dots \\ \dots & \dots & \dots \end{pmatrix}$

Note $\ker \alpha_i \oplus \mathbb{C}[H_{\alpha_i}] = \mathfrak{h}$

Effect of σ_{α_i} : $\begin{pmatrix} \dots & \dots & \dots \\ \dots & x+y & \dots \\ \dots & x-y & \dots \end{pmatrix} \mapsto \begin{pmatrix} \dots & \dots & \dots \\ \dots & x-y & \dots \\ \dots & x+y & \dots \end{pmatrix}$

i.e. it transposes $(i, i+1)$

So W acts exactly on \mathfrak{h} like our desired semisimple orbits.

Therefore:

Theorem: $\mathfrak{h}/\mathfrak{w} \longleftrightarrow \{\text{semisimple orbits}\}$

$[X] \longmapsto \mathcal{O}_X$

is a bijection.

Proof: Three things to check:

① Well-defined.

Just need: if \mathfrak{w} -conjugate, then \mathfrak{h} -ad-conjugate.

② Surjective.

Given X' semisimple, (it lies in a CSA \mathfrak{h}').

Need: X' , ~~expect \mathfrak{h}' to be~~
to be \mathfrak{h} -ad-conjugate to X .

Expect \mathfrak{h}' to be \mathfrak{h} -ad-conjugate to \mathfrak{h} .

→ Chevalley's theorem.

③ Injective.

Suppose $\mu([X_1]) = \mu([X_2])$.

$\mathcal{O}_{X_1} = \mathcal{O}_{X_2}$

① $X_1, X_2 \in \mathfrak{h}$ (if \mathfrak{h} -ad)

② $X_2 \in \mathfrak{h}$, $X_1 \in \mathfrak{g}^{X_2}$ (if \mathfrak{h} -ad, $X_1 \in \mathfrak{h}$ CSA hence abelian)

③ This is the crux of the problem, that X does not fix \mathfrak{h} and $X \cdot \mathfrak{h} \neq \mathfrak{h}$ want to make $\mathfrak{h}, X \cdot \mathfrak{h}$ further conjugate.

Philosophically: if \mathfrak{h} -ad-conjugate, then \mathfrak{w} -conjugate?

\mathfrak{w} only acts on \mathfrak{h} .

No reason to expect this if X_1, X_2 a priori only \mathfrak{h} -ad-conjugate.

As $X_1, X_2 \in \mathfrak{h}$, need them to be conjugate via sth that fixes \mathfrak{h} .

Now want to show:

④ $\mathfrak{h}, X \cdot \mathfrak{h}$ conjugate via adjoint group of \mathfrak{g}^{X_2} , say via g .

But g is inner automorphism of \mathfrak{g}^{X_2} so fixes X_2 .

$g \cdot X_1 = g \cdot X_2 = X_2$

$(g \cdot X)$ fixes \mathfrak{h} and X_1, X_2 conjugate via $(g \cdot X)$.

Just need: if \mathfrak{h} -ad-conjugate & fix \mathfrak{h} , then \mathfrak{w} -conjugate.

Key tools

3

Lemma ①: If w -conjugate, then h -ad-conjugate.

Pf: Have seen this before, (one way or another):

Suffice show for reflect² σ_α (α root.)

Inner automorphism

$$Z_\alpha = \exp(\text{ad } x_\alpha) \exp(\text{ad } (-y_\alpha)) \exp(\text{ad } x_\alpha)$$

$$Z_\alpha(h_\alpha) = -h_\alpha \quad (\text{sh. computat}^2)$$

Recall those orthogonal to h_α are

$$\ker \alpha \quad (\text{sh } \mathfrak{g} = \ker \alpha \oplus \mathbb{C}[h_\alpha])$$

Clearly $Z_\alpha(h) = h$ for $h \in \ker \alpha$ ✓

Lemma ②: Chevalley's theorem.

Various proofs:

a) Humphreys c. 16

'elementary' but very long!

b) Analytic (in ref. of text.) similar idea.

Chevalley ←

- only for \mathbb{C}

c) Algebraic geometry

Jacobson chapter IX ←

arbitrary char $k \neq 0$
 $k = \bar{k}$

- use regular elements, Zariski topology.

Chevalley's theorem

Regular elements

For $X \in \mathfrak{g}$, consider char. poly of $\text{ad } X$ (in t , say)

Degree = $\dim_k \mathfrak{g}$, const. term = $\det(\text{ad } X) = 0$ (c: $[X, X] = 0$)

$$X(\text{ad } X) = t^d + c_{d-1}(X)t^{d-1} + \dots + c_1(X)t.$$

- Coefficients c_i are homogeneous polynomials on \mathfrak{g} of deg $d-i$.

Pf: Fix basis X_1, \dots, X_d of \mathfrak{g} , then

$$X = \sum a_i X_i$$

$\text{ad } X(X_j)$ is linear combinatⁿ of a_i

so matrix for $\text{ad } X$ has entries linear combinatⁿ of a_i .

For $\phi \in \text{Aut}(\mathfrak{g})$,

$$\text{ad}(\phi(x)) = \phi \circ \text{ad} x \circ \phi^{-1}$$

so $\chi(\text{ad}(\phi(x))) = \chi(\text{ad} x)$. i.e. the case

Defⁿ: χ is $\text{Aut}(\mathfrak{g})$ -invariant.
 Min r s.t. $c_r \neq 0$ is rank of χ .

Defⁿ: $X \in \mathfrak{g}$ is regular (in textbook, regular semisimple)
 iff $c_r(X) \neq 0$.

Set of regular elements is Zariski-open and Ad -invariant.

Illustratⁿ: Generalised eigenspace decomposition w.r.t. $\text{ad} X$:

$$\mathfrak{g} = \mathfrak{g}_0^X \oplus \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda^X$$

Regular $\Leftrightarrow \dim \mathfrak{g}_0^X = \text{rk}(\mathfrak{g})$
 (else \geq)

textbook defines regular as $\dim \mathfrak{g}_0^X = \text{rk} \mathfrak{g}$.
 Can show easily that regular + semisimple = regular semisimple.

If X regular, $c_r(X) = \prod \lambda$ (I haven't been careful w sign for simplicity)

In particular if $X \in \mathfrak{h}$, then $c_r(X) = \prod \alpha(X)$ (2.1.10)

Example: \mathfrak{sl}_n .

$$\chi(\text{ad} X) = \chi(\text{ad} X_S)$$

Say X_S has eigenvalues $\lambda_1, \dots, \lambda_n$ (i.e. these are the generalised eigenvalues of X)

$$\text{then } \text{ad} X_S(E_{ij}) = (\lambda_i - \lambda_j) E_{ij}$$

$$\text{ad} X_S(E_{ii} - E_{i+1, i+1}) = 0$$

So $\text{ad} X_S$ has eigenvalues $0, \dots, 0, \lambda_i - \lambda_j$

rk. also it is hence semisimple!

$\text{rk}(\mathfrak{sl}_n) = \frac{n(n-1)}{2}$ (dim CSA of \mathfrak{sl}_n)

$$c_r(X) = \prod (\lambda_i - \lambda_j), \text{ regular iff e-values distinct cf. 2.1.14}$$

E.g. \mathfrak{sl}_2 , basis $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $X = \begin{pmatrix} & 1 \\ & \end{pmatrix}$, $Y = \begin{pmatrix} & \\ 1 & \end{pmatrix}$

$$Z = \begin{pmatrix} t & c \\ d & -t \end{pmatrix}, \chi(Z) = \lambda^2 - (t^2 + cd)$$

$$\lambda_1 = +\sqrt{t^2 + cd}, \lambda_2 = -\sqrt{t^2 + cd}, \prod (\lambda_i - \lambda_j) = -4(t^2 + cd)$$

4 Generation of CSAs

Note sln example: $X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ regular iff λ_i distinct.

Then $\mathfrak{g}^X = \{ \text{diagonal matrices} \} = \mathfrak{h}$

Else if some $\lambda_i = \lambda_j$, \mathfrak{g}^X is larger!

Lemma 2-1-9. X regular
(generation of Cartan subalgebra)

$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}} \mathfrak{g}_\lambda^X$ generalised c. space decomposition of \mathfrak{g} w.r.t. $\text{ad } X$.

Claim: $\mathfrak{h} = \mathfrak{g}_0^X$ is CSA

check:
 $\begin{bmatrix} \mathfrak{g}_\lambda^X & \mathfrak{g}_\lambda^X \\ \mathfrak{g}_\mu^X & \mathfrak{g}_\mu^X \end{bmatrix}$
 $\subset \mathfrak{g}_\lambda^X + \mathfrak{g}_\mu^X$
 So \mathfrak{h} acts blockwise
 $\begin{bmatrix} \mathfrak{h} & \mathfrak{g}_\lambda^X \\ \mathfrak{g}_\mu^X & \mathfrak{g}_\nu^X \end{bmatrix}$
 $\subset \mathfrak{g}_\lambda^X$

Pf. Consider:
 $A = \{ h \in \mathfrak{h} \mid \text{ad } h|_V \text{ non-singular} \}$
 $B = \{ h \in \mathfrak{h} \mid \text{ad } h|_V \text{ not nilpotent} \}$
 (Zariski-open)

(Zariski-open equiv to $X(\text{ad } h) \neq t^k$)
 If h not nilpotent,
 $\exists h \in B$ (Engel's thm.)

So $A \cap B \neq \emptyset$

$\text{ad } c|_V$ non-singular, $\text{ad } c|_W$ not nilpotent

\Downarrow
 $\mathfrak{g}_0^c \subsetneq \mathfrak{h}$ contradicts regularity of X
 (has minimal zero eigenspace dimension 1)

So \mathfrak{h} nilpotent

Now show $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. [X.4]

If $y \in N_{\mathfrak{g}}(\mathfrak{h})$ then $\begin{bmatrix} y & \\ & \mathfrak{h} \end{bmatrix} \in \mathfrak{g}$ want to show $y \in \mathfrak{h}$, or
 But $\text{ad } X|_{\mathfrak{h}}$ nilpotent $\Rightarrow (\text{ad } X)^M(y) = 0$
 So $\exists N$ with $\exists M$

$$0 = (\text{ad } X)^N([X, y]) = (\text{ad } X)^{N+1}(y)$$

$\Rightarrow y \in \mathfrak{g}_0^X = \mathfrak{h}$ as desired. $\Rightarrow X$ semisimple

Imp't remark: If \mathfrak{g} semisimple, then $\text{ad } X$ semisimple
 (\mathfrak{h} being in CSA \mathfrak{g}) $\Rightarrow \mathfrak{h} = \mathfrak{g}^X$

Hence textbook calls them 'regular semisimple'

Easy corollary: \mathfrak{h} CSA, if $X \in \mathfrak{h}$ regular, then $\mathfrak{h} = \mathfrak{g}^X$.
 (uniqueness of general of CSA)
 Pf: \mathfrak{h} nilpotent so $\text{ad} X|_{\mathfrak{h}}$ nilpotent so $\mathfrak{h} \subseteq \mathfrak{g}^X$, but \mathfrak{g}^X is CSA so $\mathfrak{h} = \mathfrak{g}^X$.
 (Date:)
 (\mathfrak{h} is maximal nilpotent) \square

Theorem: Any two CSAs $\mathfrak{h}_1, \mathfrak{h}_2$ are conjugate by inner automorphisms / ad i.e. $\exp(\text{ad } X)$ for $\text{ad } X$ nilpotent
 (Chevalley conjugacy of CSA)

Pf: Lemma (algebraic geom / inverse pt-thm.)
 Let $f = (f_1, \dots, f_n): \mathbb{A}^n \rightarrow \mathbb{A}^n$ be polynomial map / regular map
 (f_1, \dots, f_n)

If Jacobian $df = \left(\frac{\partial f_i}{\partial x_j} \right)$ invertible at some pt a , then $\text{im}(f)$ contains Zariski-open set.

Sketch:
 ① f_1, \dots, f_n algebraically indep. by invertibility of Jacobian.
 (Else, choose F of min deg s.t. $F(f_1, \dots, f_n) = 0$ then $\left(\frac{\partial F}{\partial y_i} (f_1, \dots, f_n) \right)$ always in kernel of Jacobian, so $\frac{\partial F}{\partial y_i} (f_1, \dots, f_n) = 0 \Rightarrow \frac{\partial F}{\partial y_i} = 0 \Rightarrow F = 0$)

② $k(x_1, \dots, x_n) / k(f_1, \dots, f_n)$ have same tr. deg. so algebraically finitely-generated so finite.

③ $S \ni g$ s.t. $k(f_1, \dots, f_n, g^{-1}) \subset k[x_1, \dots, x_n, g^{-1}]$ integral i.e. corresponds to finite morphism into $\{g \neq 0\}$

But finite morphism is surjective! So $\text{im}(f)$ contains $\{g \neq 0\}$.

Now fix any CSA \mathfrak{h} .

Key step: Consider conjugated map:

$$f: \mathfrak{g} \rightarrow \mathfrak{g}$$

$$h + \sum x_j b_j \mapsto e^{x_1 \text{ad} b_1} \dots e^{x_n \text{ad} b_n} (h)$$

Basis for each root space \mathfrak{g}_α .
 (if \mathfrak{h} only nilpotent, still have generalised root space decomposition)
 $\det \mathfrak{h}^*$ not

<

Remark: In analytic proof, also consider conjugate map

$$\varphi : G \times G \rightarrow G$$

$$(x, H) \mapsto x \cdot H$$

consider its differential $d\varphi$!

Now, consider Jacobian/differential of f at a pt a

In direct² $h + b$, this is $\sum x_j a_j b_j$

$$\frac{d}{dt} f(a + t(h+b)) \Big|_{t=0}$$

expand to can ignore all t^2 and above terms

$$\prod (I + t x_j a_j b_j) (a + t h)$$

So derivative is coefficient of linear term t

$$= h + \sum x_j a_j b_j (a)$$

$$= h + [b, a] (= h - [a, b])$$

Remark: Computat² is similar in the analytic case!

\Rightarrow differential of f , df acts as id on \mathfrak{g} and $-ad a$ on root spaces \mathfrak{g}_α

eigenvalues $\alpha(a)$

so implies $\alpha(a) \neq 0 \forall$ roots α .

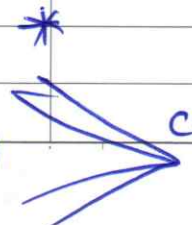
But only finitely many roots α_i each $\alpha_i(a) \neq 0$ is a Zariski-open condition!

To conclude: for any two CSAs $\mathfrak{h}_1, \mathfrak{h}_2$ w corresponding conjugate maps f_1, f_2 , their image contains Zariski-open, and regular elements are Zariski-open, so \exists regular $x, x_1, x_2 \in G_{ad}, h_1, h_2 \in \mathfrak{h}_1, \mathfrak{h}_2$ s.t.

$$x = x_1 h_1 = x_2 h_2$$

$\mathfrak{h}_1, \mathfrak{h}_2$ conjugate via x_1, x_2 to \mathfrak{g}^x , hence to each other

Cont on last pg
7



unique CSA \mathfrak{g}^x !

Back to:

Lemma 3: If G -conjugate ξ fix h , then W -conjugate.
 pk: Algebraic proof (for arbitrary k) in Bombalei Chap 8 Sect 5
 → elementary but v. long!

Analytic proof: use covering group \tilde{G} of G and f.d. rep!

First: Suppose automorphism $x \in \text{Gal}$ leaves h invariant.

x acts on roots μ by $x \cdot \mu = (h \mapsto \mu(x^{-1}h))$ must send simple roots (base) Δ to another base of simple roots, $x \cdot \Delta$.

$x \cdot \mu = (h \mapsto \mu(x^{-1}h))$

positive roots to positive roots. another system of

$x \cdot \rho = \rho \cdot x$

But Weyl group W acts faithfully on bases,

$\exists w \in W, w \cdot \Delta = x \cdot \Delta$

And by lemma 1, $\exists y \in \text{Gal}$ with same act^k as $w \in W$

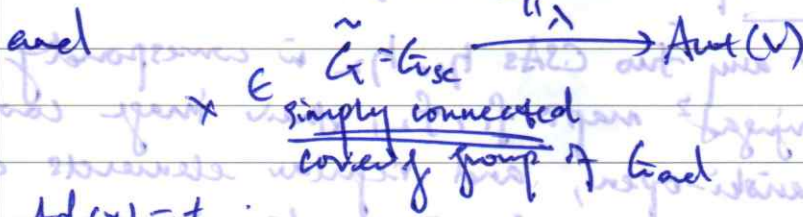
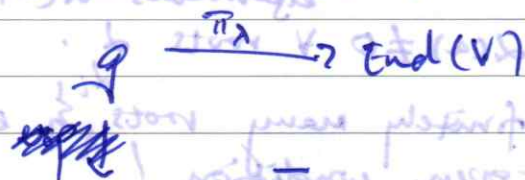
So $y \cdot x$ leaves Δ invariant.

Want to show $x \cdot y$ have same act^k on h , then w is the desired.

So suffice show: if $t \in \text{Gal}$ leaves Δ invariant, then $t|_h$ is identity.

Suffice show: $t \cdot \lambda = \lambda \quad \forall$ dominant integral λ (choose usual basis for h)

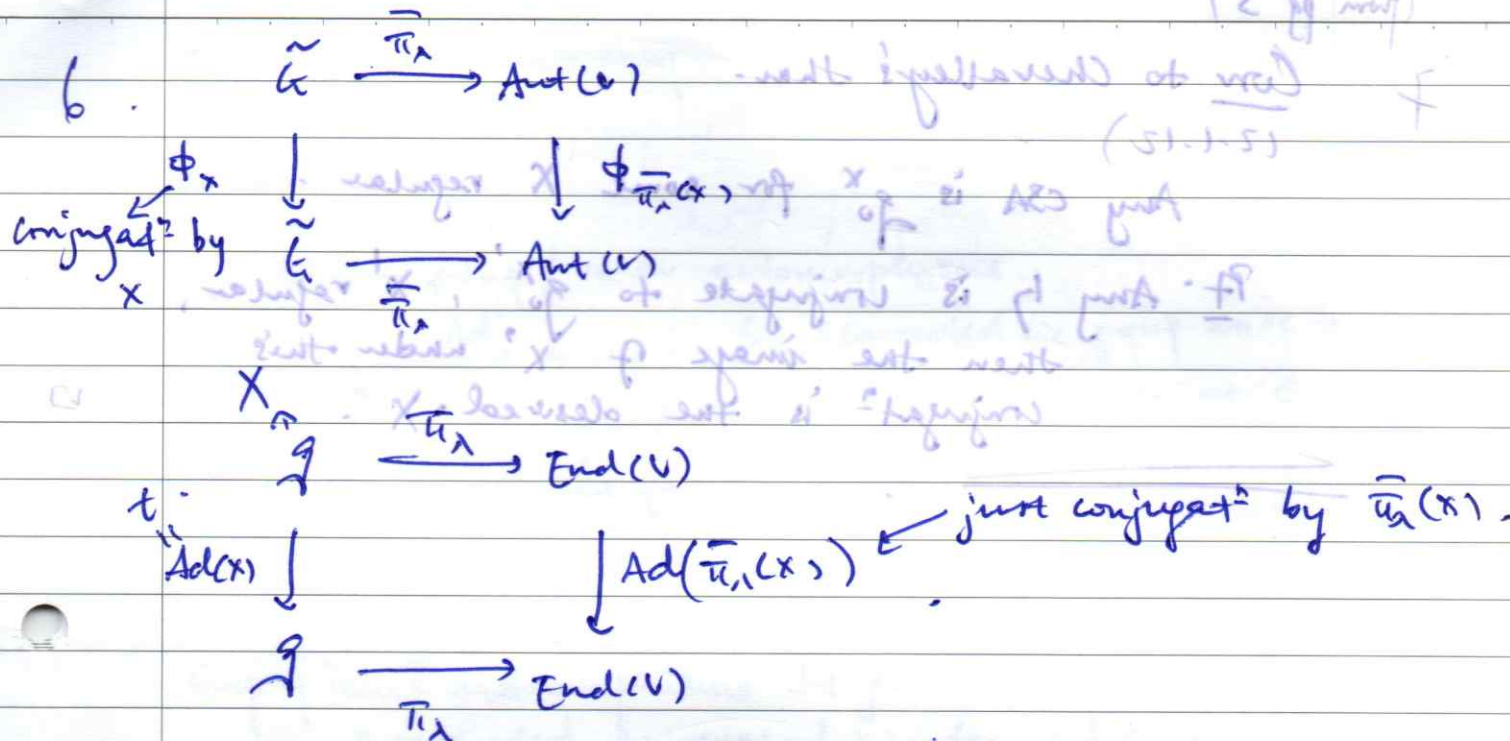
Key idea: consider irred. f.d. rep of g in highest weight λ !!



G.S.C. so can 'lift' or 'integrate' to a Lie group map $\tilde{\pi}_\lambda$

$\text{Ad}(x) = t$

(2 pg max)



$$\pi_\lambda(t \cdot X) = \bar{\pi}_\lambda(x) \pi_\lambda(X) \bar{\pi}_\lambda(x)^{-1} \Rightarrow \pi_\lambda(t \cdot X) \bar{\pi}_\lambda(x) = \bar{\pi}_\lambda(x) \pi_\lambda(X)$$

if $v \in V_\mu$ (weight space of V)

then

$$\begin{aligned}
 \pi_\lambda(t \cdot X) \bar{\pi}_\lambda(x)(v) &= \bar{\pi}_\lambda(x) \pi_\lambda(X)v \\
 &= \mu(H) \bar{\pi}_\lambda(x)(v)
 \end{aligned}$$

$$\Rightarrow \bar{\pi}_\lambda(x)(v) \in V_{t \cdot \mu}$$

so $\forall \mu, V_\mu \cong V_{t \cdot \mu}$ via $\bar{\pi}_\lambda(x)$! (1)

Consider highest weight ^{/maximal} vector $v^+ \in V_\lambda$.
 But now t fixes simple, positive roots so fixes the Borel subalgebra \mathfrak{b} (which kills v^+)

$$\begin{aligned}
 \Rightarrow \bar{\pi}_\lambda(x)v^+ &\text{ is still killed by all } \mathfrak{b} \\
 \Rightarrow &\text{ it is still maximal} \\
 \Rightarrow V_\lambda &\cong V_\lambda \text{ via } \bar{\pi}_\lambda(x) \text{ ! (2)}
 \end{aligned}$$

(1) + (2) : $\lambda = t \cdot \lambda$ as desired.

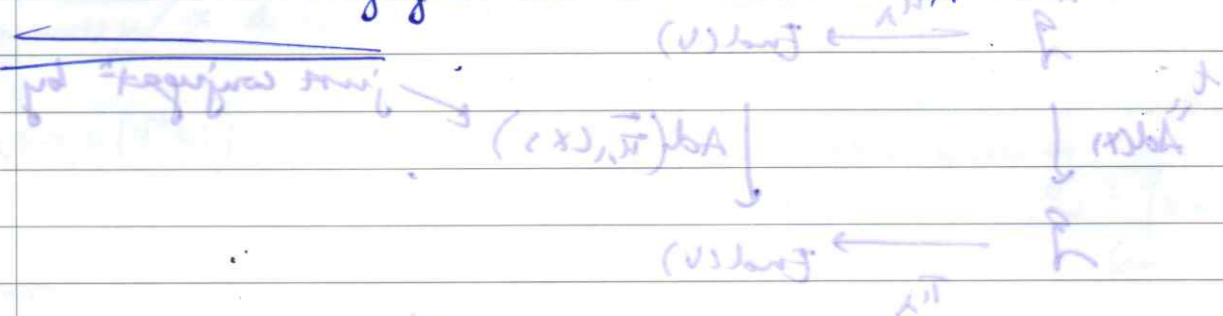
(from pg 5)

7

Conv to Chevalley's thm.
(2.1.12)

Any CSA is g_0^X for some X regular

Pf. Any h is conjugate to $g_0^{X'}$, X' regular, then the image of X' under this conjugatⁿ is the desired X . \square



$$\pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} \pi_{\mathfrak{h}}(t)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

$$\pi_{\mathfrak{h}}(X)_{\mathfrak{h}} \pi_{\mathfrak{h}}(t)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

if $v \in V$ (weight space of V)

$$v \left(\frac{H}{2} \right)_{\mathfrak{h}} \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} = v \left(\frac{H}{2} \right)_{\mathfrak{h}} \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

$$v \left(\frac{H}{2} \right)_{\mathfrak{h}} \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} = v \left(\frac{H}{2} \right)_{\mathfrak{h}} \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

$$\pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

Consider highest weight vector $v \in V^+$.
But none of these weights, positive roots so from the lower subalgebra \mathfrak{p} (lowest roots v^+)

$$\pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

$$\pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$

$$\pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X)_{\mathfrak{h}} = \pi_{\mathfrak{h}}(v) \pi_{\mathfrak{h}}(X \cdot t)_{\mathfrak{h}}$$