Sheaves

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Outline

- Prologue: Motivating example - structure sheaf of variety
- Definitions - presheaf, sheaf, stalk
- (Abelian) category of sheaves on X
- Sheaves on different spaces
- Flasque sheaves, glueing sheaves, sheaves on basis
- Epilogue: back to motivating example

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- Local properties: given a rational function, being regular at a point is a local property. Should be able to recover a regular function (and uniquely) from its restriction on smaller opens [*sheaf*]
- Local ring: We have seen the importance of the local ring O_P of functions regular at a point P. We identify two such functions if they coincide on an open neighbourhood of P. Conversely, any function regular at P is regular in an open neighbourhood of P. [*stalk*]

Presheaf

Definition (Presheaf)

Given a topological space X, a **presheaf** $\mathscr{F}$ of abelian groups (resp. rings, etc.) on X consists of the following defining data:

1. For every open $U \subseteq X$, an abelian group $\mathscr{F}(U)$.
   The elements of $\mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ over U, and it is common to use $\Gamma(U, \mathscr{F})$ to denote $\mathscr{F}(U)$.
2. For every inclusion $V \subseteq U$ a '*restriction*' group morphism $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$.
   In keeping with the notion of 'restriction', we often write $s|_V$ for $\rho_{UV}(s)$.

Furthermore, we stipulate:

- $\mathscr{F}(\emptyset) = 0$
- $\rho_{UU} = \mathrm{id}_{\mathscr{F}(U)}$
- (*Compatibility of restrictions*) For any three opens $W \subseteq V \subseteq U$, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Sheaf

Definition (Sheaf)

A presheaf $\mathscr{F}$ (on a topological space X) is called a **sheaf** if it in addition satisfies:

- (*'Glueing property'*) For each open U, open covering $\{V_i\}$ of U, and sections s_i over V_i such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j (compatibility), then there is a **unique** section s over U such that $s|_{V_i} = s_i$ for all i.

Remark

Hartshorne stipulates two conditions (3) and (4), but (3) is just the uniqueness.

Stalk

Definition (Stalk)

Given any presheaf $\mathscr{F}$ on X and a point P of X, the **stalk** $\mathscr{F}_P$ is the direct limit of the groups $\mathscr{F}(U)$ for all open sets containing P, via the restriction maps ρ.

In other words, an element of $\mathscr{F}_P$ is an equivalence class of pairs (U, s) (where U is open and s a section over U), and two pairs $(U, s), (V, t)$ are equivalent iff s, t agree on an open neighbourhood W of $P \subseteq U \cap V$ (via the restriction ρ), i.e. $s|_W = t|_W$.

The elements of stalk are often called **germs**.

What to use as sections?

Remark

In what follows, we will often be using sheaves of rings, also known as *ringed spaces*, or *locally ringed spaces* if the stalks at each point are local rings.

However, for simplicity's sake, since the definitions and concepts carry through almost verbatim from abelian groups to rings, in order to focus on sheaf theory first (and in keeping with Hartshorne) we have phrased everything in terms of abelian groups.

Indeed, later we will even define sheaves of modules over ringed spaces. At its heart, the basic 'local in nature' ideas are all the same. Hence why it is important to start with a treatment of sheaf theory.

Category of sheaves

Of course, as always, in order to be meaningful we want to have notions of *morphisms* between (pre)sheaves, so that we may speak of a *category*.

There is really only one definition that makes sense:

Definition (Morphism of (pre)sheaves)

Given two (pre)sheaves $\mathscr{F}, \mathscr{G}$ (on a fixed topological space X), a *morphism* of (pre)sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is, for each open U, a morphism of groups $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ which is *compatible with the respective (pre)sheaf structures*, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ \downarrow{\rho_{UV}} & & \downarrow{\rho_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \end{array}$$

Any morphism ϕ clearly induces a well-defined morphism $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ on stalks.

Presheaf vs sheaf

Already from the definition of morphisms, we can see that if we are to work consistently with sheaves (i.e. local property) where our general definitions only involve presheaf notions, we need a way of ensuring or at least a canonical way of turning presheaves into sheaves.

Indeed, there is a universal way to do so.

Sheafification

Definition (*'Sheafification'*)

Given any presheaf $\mathscr{F}$, there exists a sheaf $\mathscr{F}^+$ and morphism $\theta : \mathscr{F} \to \mathscr{F}^+$, unique up to unique isomorphism, satisfying the following *universal property*: for any sheaf $\mathscr{G}$ and any morphism $\phi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : \mathscr{F}^+ \to \mathscr{G}$ such that $\phi = \psi \circ \theta$.

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\forall \phi} & \mathscr{G} \\
\downarrow \theta & \nearrow \exists! \psi & \\
\mathscr{F}^+ & &
\end{array}
$$

Proof.

Uniqueness up to unique isomorphism follows from straightforward general nonsense around the universal property. So we focus on existence, which is in fact conceptually important.

Sheafification

Proof.

The construction is really rather intuitive if we think from the perspective of sheaves of functions, and about what it means for a presheaf to fail to be a sheaf.

The most common reason is that there are local sections which fail to glue together to give a global section, because the sections in our presheaf are defined insufficiently locally. (We will see this explicitly in our example next.)

Sheafification

Proof.

Each section s over an open U already corresponds to a germ (U, s) at each point $P \in U$.

Therefore, in order to be sufficiently local, we let:

- $\mathscr{F}^+(U)$ be the set of all functions on U, sending each point P to a germ in $\mathscr{F}_P$, such that the function is *'locally defined by a local section'*: that is, at each point P, there is a neighbourhood $V \subseteq U$ of P and a local section t of V such that $s(Q) = (V, t) \in \mathscr{F}_Q$ for all $Q \in V$.
- The restriction maps in $\mathscr{F}^+$ are natural restriction of functions;
- We have already seen the accompanying morphism θ: it sends each s to the function which is defined globally over U by s. ((U, s) at each point P)

Sheafification

Proof.

The verification that $\mathscr{F}^+$ is a sheaf is then a completely routine exercise in local properties: given local sections(functions) of $\mathscr{F}^+$ there is clearly only one way to glue them together into a global section(function), and the resulting global section(function) must clearly also be locally defined by local section.

Sheafification

Proof.

Furthermore, if $\mathscr{F}$ is already a sheaf, then it is plain to see that $\mathscr{F}, \mathscr{F}^+$ are isomorphic via θ: local sections of $\mathscr{F}$ can be patched together uniquely to give a global section on any open U, so a function locally defined by local sections is uniquely globally defined by a global section (which is precisely the image of θ).

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Finally, to verify the universal property, all we need to do is exhibit a unique morphism $\psi : \mathscr{F}^+ \to \mathscr{G}^+$. But since $\phi : \mathscr{F} \to \mathscr{G}$ already tells us where all the sections of $\mathscr{F}$ are sent, we can do nothing but send the locally defining local sections of each function(section) in $\mathscr{F}^+$ to the corresponding local section of $\mathscr{G}$, giving a function(section) in $\mathscr{G}^+$. □

Sheafification

Remark

Remark also that even if $\mathscr{F}$ is not sheaf, it is clear to see why $\mathscr{F}_P = \mathscr{F}_P^+$ via θ, since sections of $\mathscr{F}^+$ are locally defined at P by local sections of $\mathscr{F}$. We will use this later when talking about stalks.

Remark

Also, sheafification is in some sense functorial, that is, given a morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$, there is a morphism $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ (again, send locally defining local sections to their image) commuting with the respective θs.

Sheafification - example

Example (Hartshorne Exercise 1.1)

For each abelian group A, consider the *constant presheaf* $\mathscr{F}$ on X defined by $\mathscr{F}(U) = A$ for all $U \neq \emptyset$, with all restriction maps the identity. In general, this is *not* a sheaf: sections on say two disjoint opens cannot always be patched together to give a global section (which must always be constant).

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Therefore, the sheafification of $\mathscr{F}$ is the sheaf of *locally constant* functions, i.e. locally defined by a constant function(section of $\mathscr{F}$). In other words, the sections of $\mathscr{F}^+$ are the functions with each fibre being open, equivalently continuous functions if A is given the discrete topology.

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Remark

One minor example of note: for X being (irreducible) variety, the constant presheaf on X associated to the rational functions $k(X)$ is immediately a sheaf (cf. also part of Hartshorne Ex 1.21). The sheaf associated to divisor is a subsheaf of this sheaf, and so too is the sheaf of regular functions!

Sheafification

Remark (Structure sheaf)

Later on, when defining the structure sheaf of an arbitrary Spec A, we will use a similar notion of locally defined functions. This is the right notion that is captured by a *sheaf*.

Indeed, the structure sheaf $\mathscr{O}$ of Spec A can be fairly naturally completely described in terms of sheafification. All we do is specify $\mathscr{O}(U) = A_S$, the localisation by $S := A \setminus \cup_{\mathfrak{p} \in U} \mathfrak{p}$, then sheafify.

Remark (Sheaf associated to module)

Similar remark applies when we consider quasi-coherent sheaves later, i.e. when constructing the sheaf associated to a given A-module M.

Remark ('Sheafification' of sheaf on basis)

We will see how to further refine this later by considering only basis open sets.

Category of sheaves as abelian category

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- For each morphism $\phi : \mathscr{F} \to \mathscr{G}$, there are obvious ways to define its image, kernel and cokernel: $(\text{im } \phi)(U) = \text{im } (\phi(U))$, $(\ker \phi)(U) = \ker (\phi(U))$, $(\text{coker } \phi)(U) = \text{coker } (\phi(U))$. (Check that they are well-defined presheaves in the natural way!)

Kernel, injective

- Now, if $\mathscr{F}, \mathscr{G}$ are sheaves, then $\ker \phi$ is also a sheaf: local sections in $\mathscr{F}$ which are killed by ϕ can be patched (uniquely) to a global section of $\mathscr{F}$, which must be killed by ϕ (by the sheaf property of $\mathscr{G}$!). So we just define ker as it is.

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Definition (Kernel, injective morphism)

The kernel of a morphism is just the plain kernel presheaf, which is automatically a sheaf. An injective morphism is one with $\ker \phi = 0$, which is iff each section morphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

Image

- Unfortunately, im ϕ and coker ϕ need not be sheaves. (E.g. for im , intuitively local sections in im ϕ already glue uniquely to a global section in $\mathscr{G}$, but this global section need not lie in the image because we can get no guarantees on the preimages in $\mathscr{F}$.) Therefore

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Definition (Image, cokernel)

Given any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, im ϕ and coker ϕ are defined as the sheafification of the image presheaf and cokernel presheaf respectively.

Image as subsheaf

Lemma (Identifying image with subsheaf of codomain; Hartshorne Exercise 1.4)

Given $\phi : \mathscr{F} \to \mathscr{G}$, im ϕ may be identified naturally as a subsheaf of $\mathscr{G}$.

Proof.

The inclusion morphism $\iota : \phi(\mathscr{F}) \hookrightarrow \mathscr{G}$ (here $\phi(\mathscr{F})$ is just the plain image presheaf) lifts by the universal property to a morphism $\psi : \text{im } \phi \to \mathscr{G}$. So, just show that ψ is injective. For this, recall how ψ is defined. If a function(section of im ϕ), locally defined by local sections of $\phi(\mathscr{F})$, is sent to 0, then the local sections must also be sent to 0 by ι, and hence are each 0, but that means the function is itself 0. □

Remark

Since the sheafification morphism $\theta : \phi(\mathscr{F}) \to \text{im } \phi$ is therefore also injective, we thus have 'inclusions' of sub(pre)sheaves $\phi(\mathscr{F}) \subseteq \text{im } \phi \subseteq \mathscr{G}$.

Surjective

Definition (Surjective)

A morphism $\phi : \mathscr{F} \to \mathscr{G}$ is *surjective* if im $\phi = \mathscr{G}$.

Remark (Necc. and suff. condition for surjectivity; Hartshorne Exercise 1.3)

From the previous Lemma on the identification of im ϕ as a subsheaf of $\mathscr{G}$, we get almost for free the following criterion for a morphism ϕ to be surjective:
it is iff every section s of $\mathscr{G}$ over every open U can be given by patching together local sections $\phi(t_i)$ in the plain image $\phi(\mathscr{F})$ ($t_i \in \mathscr{F}(U_i)$), i.e. $s|_{U_i} = \phi(t_i)$.

Isomorphism = injective+surjective

Lemma (Isomorphism = injective+surjective; Hartshorne Exercise 1.5)

A morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is an isomorphism iff it is injective and surjective.

Proof.

Injective + surjective $\implies$ isomorphism: Use injectivity first. We can identify $\mathscr{F}$ with a subsheaf of $\mathscr{G}$. Then surjectivity means the section maps $\phi(U)$ are also surjective, so isomorphism follows.

Isomorphism $\implies$ injective + surjective: The section maps $\phi(U)$ must be injective, so ϕ injective and again identify $\mathscr{F}$ with a subsheaf of $\mathscr{G}$. Now the section maps $\phi(U)$ must also be surjective, so ϕ surjective. □

Exact sequences

Definition (Exact sequences)

Exact sequences of sheaves are defined in completely the usual way (for abelian categories).

Local nature of sheaf

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So we expect the stalks of sheaves to carry a great deal of information.

In doing so, we can encapsulate arguments of a local nature by simply considering stalks.

Local nature of sheaf

Lemma (*"Stalks and morphisms commute"*; Hartshorne Exercise 1.2)

For any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, we have $(\ker \phi)_P = \ker(\phi_P)$, $(\mathrm{im}\ \phi)_P = \mathrm{im}\ (\phi_P)$.

Proof.

For ker :
$(\ker \phi)_P \subseteq \ker(\phi_P)$: a germ of sections killed by ϕ must be killed by ϕ_P;
$\ker(\phi_P) \subseteq (\ker \phi)_P$: conversely, a germ killed by ϕ_P is in a small enough neighbourhood zero in $\mathscr{G}$ i.e. a section killed by ϕ, hence a germ of sections killed by ϕ.

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For im : Same, but we work with the plain image presheaf $\phi(\mathscr{F})$ and use previous remark that $(\mathrm{im}\, \phi)_P = (\phi(\mathscr{F}))_P$. □

Local nature of sheaf

Lemma

Let $\mathscr{F}$ be a subsheaf of $\mathscr{G}$. Then $\mathscr{F} = \mathscr{G}$ iff $\mathscr{F}_P = \mathscr{G}_P$ for all P.

Proof.

Suppose $\mathscr{F}_P = \mathscr{G}_P$. Any section $s \in \mathscr{G}(U)$ gives rise to germs $(U, s) \in \mathscr{G}_P$ for each $P \in U$. These germs must also live in $\mathscr{F}_P$ so correspond to local sections of $\mathscr{F}$ which can be patched to give a global section $t \in \mathscr{F}(U)$. But by uniqueness, we must have $t = s$. So $\mathscr{F}(U) = \mathscr{G}(U)$ as desired. The converse is trivial. □

Local nature of sheaf

Corollary (*Stalks carry sheaf data*; Hartshorne Exercise 1.2 cf. Prop 1.1)

ϕ is (resp. injective, surjective, isomorphism) iff ϕ_P is (resp. injective, surjective, isomorphism) for all P.
A sequence of sheaves and morphisms is exact iff the corresponding sequence of stalks and morphisms of stalks is exact for all P.

Proof.

Recall previous lemma on commuting of stalks and morphisms.

□

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Recall previous lemma on commuting of stalks and morphisms.
Injective: ϕ_P injective $\iff$ $\ker \phi_P = 0$ $\iff$ $(\ker \phi)_P = 0$ for all P $\iff$ $\ker \phi = 0$ (by previous lemma and viewing zero sheaf as subsheaf of $\ker \phi$) $\iff$ ϕ injective.

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Recall previous lemma on commuting of stalks and morphisms.
Injective: ϕ_P injective $\iff$ ker $\phi_P = 0 \iff (\ker \phi)_P = 0$ for all P $\iff$ ker $\phi = 0$ (by previous lemma and viewing zero sheaf as subsheaf of ker ϕ) $\iff$ ϕ injective.
Surjective: ϕ_P surjective $\iff$ im $\phi_P = \mathscr{G}_P \iff (\text{im } \phi)_P = \mathscr{G}_P \iff$ im $\phi = \mathscr{G}$ (by previous lemma and viewing im ϕ as subsheaf of $\mathscr{G}$) $\iff$ ϕ surjective. □

Quotient

Definition (Quotient sheaf)

If $\mathscr{F}'$ is subsheaf of sheaf $\mathscr{F}$, the *quotient sheaf* is defined as the sheafification of the presheaf with sections given by $\mathscr{F}(U)/\mathscr{F}'(U)$ for each open U (and the natural inherited restriction maps).

By earlier remark that sheafification preserves stalks, we have $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$.

Isomorphism theorems

Lemma ('First isomorphism theorem 1'; Hartshorne Exercise 1.7a)

By construction, we immediately see that im $\phi \cong \mathscr{F}/\ker\phi$ *in a natural way. (Both sides are sheafification of the same plain quotient presheaf.)*

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By construction, we immediately see that im $\phi \cong \mathscr{F}/\ker\phi$ in a natural way. (Both sides are sheafification of the same plain quotient presheaf.)

However, as remarked earlier, dealing with im is less straightforward, and we should encapsulate local arguments by taking stalks.

Lemma ('First isomorphism theorem 2'; Hartshorne Exercise 1.7b)

For $\phi : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves, we have coker $\phi \cong \mathscr{F}/\text{im }\phi$.

Isomorphism theorems

Proof.

Denote by $\phi(\mathscr{F})$ and $(\mathscr{G}/\phi(\mathscr{F}))_{pre}$ the plain image and cokernel presheaf respectively. As before, we identify im ϕ as subsheaf of $\mathscr{G}$ via an injective morphism, so the induced morphism of stalks is injective, with image equal to $(\operatorname{im}\phi)_P = \phi(\mathscr{F})_P$ (since sheafification preserves stalks).

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On the other hand we have the natural quotient map $\mathscr{G} \to (\mathscr{G}/\phi(\mathscr{F}))_{pre} \underset{\theta}{\to} \text{coker } \phi$ and (again since sheafification preserves stalks) the induced morphism of stalks is surjective, with kernel equal to $\phi(\mathscr{F})_P$.

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On the other hand we have the natural quotient map $\mathscr{G} \to (\mathscr{G}/\phi(\mathscr{F}))_{pre} \underset{\theta}{\to}$ coker ϕ and (again since sheafification preserves stalks) the induced morphism of stalks is surjective, with kernel equal to $\phi(\mathscr{F})_P$.

So we have a sequence $0 \to \text{im } \phi \to \mathscr{G} \to \text{coker } \phi \to 0$ with corresponding sequence on stalks being short exact, so by previous result, this sequence is short exact. In particular we can identify im ϕ as a subsheaf of $\mathscr{G}$ and im ϕ is the kernel of the surjective morphism $\mathscr{G} \to \text{coker } \phi$, so by previous isomorphism theorem we have $\text{coker } \phi \cong \mathscr{F}/\text{im } \phi$. □

Short exact sequence

Remark ('Short exact sequence'; Hartshorne Exercise 1.6)

Similarly: we have a short exact $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ iff we can identify $\mathscr{F}'$ with a subsheaf of $\mathscr{F}$ and $\mathscr{F}''$ with the quotient $\mathscr{F}/\mathscr{F}'$.

In interest of time, we omit the verification (it is very similar to previous).

Taking sections

Remark (Left exactness of 'taking sections' functor; Hartshorne Exercise 1.8)

As already remarked, ϕ is injective iff $\phi(U)$ is. We immediately obtain the following: if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact, then the induced sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$ is exact for all open U.
But ϕ surjective and $\phi(U)$ surjective have no relation. So we do not get right exactness.

Direct sum and limits

Remark (Direct sum, direct limit, inverse limit; Hartshorne Exercise 1.9, 10, 12)

In similar spirit we can define the direct sum, direct limit, inverse limit of sheaves which are indeed direct sum, direct limit, inverse limit in the category of sheaves on X.

The verifications are all completely routine. For example, for direct sum we just have to verify glueing axiom, which we can do on the direct summands.

Hom-sets in category of sheaves

Our final order of business in the category of sheaves on X is to show there is a natural way to view each 'hom-set' as itself a sheaf.

First observe for each open U there is an obvious restriction $\mathscr{F}|_U$ of a sheaf $\mathscr{F}$ to U with the induced subspace topology.

Lemma (Sheaf Hom; Hartshorne Exercise 1.15)

Let $\mathscr{F}, \mathscr{G}$ be sheaves (of abelian groups) on X. For each open U there is a natural abelian group structure on the morphisms between restricted sheaves $Hom_X(\mathscr{F}|_U, \mathscr{G}|_U)$, and the presheaf $Hom(\mathscr{F}, \mathscr{G})$ defined by $Hom(\mathscr{F}, \mathscr{G})(U) = Hom_X(\mathscr{F}|_U, \mathscr{G}|_U)$ (with the natural restriction maps) is a sheaf called the sheaf of local morphisms *(of $\mathscr{F}$ to $\mathscr{G}$).*

Hom-sets in category of sheaves

Proof.

Omitting the routine verifications, the real content is this. Suppose we are given local morphisms on an open cover $\{U_i\}$ of U. Then for each open $V \subseteq U$ and section $s \in \mathscr{F}(V)$, it has local sections $s|_{V \cap U_i}$ which are sent to local sections of $\mathscr{G}$, which must glue uniquely to a global section $t \in \mathscr{G}(V)$. This is where s must be sent by our global morphism. So local morphisms glue uniquely to global morphism. $\square$

Sheaves on different topological spaces

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- Already when we consider the restriction $\mathscr{F}|_U$, we see that we need a more systematic way to consider sheaves on different topo. spaces.
- The starting point is clearly naturally a continuous map $f : X \to Y$ of topo. spaces.

Sheaves on different topological spaces

Definition (Direct image)

Given sheaf $\mathscr{F}$ on X, the *direct image* sheaf $f_*\mathscr{F}$ on Y is defined by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ for each open $V \subseteq Y$.

Sheaves on different topological spaces

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Proof.

Check that $f_*\mathscr{F}$ is sheaf: local sections of $f_*\mathscr{F}$ correspond to preimage local sections of $\mathscr{F}$ which can be glued uniquely to a global section. □

Sheaves on different topological spaces

Example (Skyscraper sheaf; Hartshorne Exercise 1.17)

Let X be a space and P a point, fix an abelian group A and consider the constant sheaf $\mathscr{C}$ on $\overline{\{P\}}$ (check this is sheaf: because every open contains P).

Now the inclusion $\iota : \overline{\{P\}} \to X$ induces a direct image sheaf $\iota_*\mathscr{C} =: \mathscr{F}$. From definitions we see $\mathscr{F}(U) = A$ iff $P \in U$ and 0 otherwise, with the obvious identity/zero restriction maps respectively. This is called the *skyscraper sheaf* at P.

Sheaves on different topological spaces

In the other direction, the situation is made complicated by two things: image of open is not always open, and we are not able to guarantee that the presheaf constructed in the natural way is a sheaf (heuristically, same reason as why we cannot guarantee image presheaf is sheaf).

Sheaves on different topological spaces

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Definition (Inverse image)

Given sheaf $\mathscr{G}$ on Y, the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X is defined by the *sheafification* of the presheaf with sections $\varinjlim_{f(U)\subseteq V} \mathscr{G}(V)$ for each open $U \subseteq X$.

Sheaves on different topological spaces

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Example (Inverse image and stalks)

If f is inclusion of a single point P into Y, then $f^{-1}\mathscr{G}$ is nothing but the stalk $\mathscr{G}_P$.

Indeed, inverse images 'preserve' stalks: we have naturally $(f^{-1}\mathscr{G})_P = \mathscr{G}_{f(P)}$. This is fairly immediate from the definition (again use that sheafification preserves stalks).

Sheaves on different topological spaces

Example (Restriction)

If f is inclusion of open U (with induced topo.) into Y, then f sends open to open so $f^{-1}\mathscr{G}$ is nothing but the obvious restriction $\mathscr{G}|_U$ from earlier.

Sheaves on different topological spaces

Example (Restriction)

If f is inclusion of open U (with induced topo.) into Y, then f sends open to open so $f^{-1}\mathscr{G}$ is nothing but the obvious restriction $\mathscr{G}|_U$ from earlier.

Now in general for f being inclusion of subset X (with induced topo.) into Y, we can define the restriction sheaf $\mathscr{G}|_X := f^{-1}\mathscr{G}$.

Observe that since inverse image preserves stalks, restriction also preserves stalks.

Sheaves on different topological spaces

A natural ‘converse’ to restriction is to extend a sheaf on a subset to the whole space.

Example (Extending sheaf by zero; Hartshorne Exercise 1.19)

Suppose X is a space, Z closed and $U = X \setminus Z$ open, both with the induced topology and with inclusions $i : Z \to X, j : U \to X$.

If $\mathscr{F}$ is a sheaf on Z, the sheaf $i_*\mathscr{F}$ is called the sheaf obtained by *extending* $\mathscr{F}$ *by zero outside* Z, and we often just view $\mathscr{F}$ as a sheaf on X. One easily checks $(i_*\mathscr{F})_P = \mathscr{F}_P$ in Z, and 0 outside Z.

Sheaves on different topological spaces

Example (Extending sheaf by zero; Hartshorne Exercise 1.19)

If $\mathscr{F}$ is a sheaf on U however, the above is not desirable as there may be non-zero stalks outside U.

Instead, we let $j_!\mathscr{F}$ be the *sheafification* of the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$ and 0 otherwise. This is called the sheaf obtained by *extending $\mathscr{F}$ by zero outside U*. Again, one easily checks $(i_*\mathscr{F})_P = \mathscr{F}_P$ in U, and 0 outside U (sheafification preserves stalks).

Adjointness of direct, inverse image

The definition of inverse image may seem slightly unnatural and is indeed slightly difficult to work with in practice. However, we will now show that the inverse image is in fact the right natural definition in the following sense.

First observe that f_* and f^{-1} are functorial in the sense that they can also be extended naturally to morphisms of sheaves (here for the inverse image one may also recall that sheafification is functorial).

Adjointness of direct, inverse image

Proposition (Adjoint property of direct, inverse image; Hartshorne Exercise 1.18)

There is a natural bijection

$$Hom_X(f^{-1}\mathscr{G}, \mathscr{F}) = Hom_Y(\mathscr{G}, f_*\mathscr{F})$$

making f_* and f^{-1} adjoint functors.

Proof.

We focus on the conceptually important parts, leaving the rest to general nonsense.

Adjointness of direct, inverse image

Proof.

Given a sheaf morphism $\psi : \mathscr{G} \to f_*\mathscr{F}$, for each open $V \subseteq Y$ we have the induced morphism $\psi(V) : \mathscr{G}(V) \to f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$.

For each open $U \subseteq X$ and $f(U) \subseteq V$ we have $U \subseteq f^{-1}(V)$.

So now the picture is clear: via the restriction maps, we take the limit of $\psi(V)$ over $f(U) \subseteq V$:

$$f^{-1}\mathscr{G}(U) = \varinjlim_{f(U) \subseteq V} \psi(V) \to \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$$

and together with universal property of sheafification this gives a morphism $f^{-1}\mathscr{G} \to \mathscr{F}$.

Adjointness of direct, inverse image

Proof.

The converse is now clear: suppose given a morphism $\phi : f^{-1}\mathscr{G} \to \mathscr{F}$, we want to get back to a morphism $\mathscr{G} \to f_*\mathscr{F}$.

But since we defined earlier direction via limit of maps, so for each open $V \subseteq Y$ we can only send $\mathscr{G}(V)$ via its inclusion in the limit $\varinjlim_{f(f^{-1}(V))\subseteq W} \mathscr{G}(W)$, i.e.

$$\mathscr{G}(V) \to \varinjlim_{f(f^{-1}(V))\subseteq W} \mathscr{G}(W) = f^{-1}\mathscr{G}(f^{-1}(V)) \to \mathscr{F}(f^{-1}(V)) = f_*\mathscr{F}(V)$$

□

Adjointness of direct, inverse image

Remark

When $\mathscr{F} = f^{-1}\mathscr{G}$, we get a natural bijection

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, f^{-1}\mathscr{G}) = \mathrm{Hom}_Y(\mathscr{G}, f_*f^{-1}\mathscr{G}),$$

so id on the LHS corresponds to a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$ (this is standard general nonsense around adjoints, or we can also write down the map explicitly based on the previous).

Similarly, we have a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

Adjointness of direct, inverse image

Remark (Extending sheaf by zero, Hartshorne Exercise 1.19)

As before, suppose X is a space, Z closed and $U = X \setminus Z$ open, both with the induced topology and with inclusions $i : Z \to X, j : U \to X$.

If $\mathscr{F}$ is a sheaf on X, then $j_!\mathscr{F}|_U$ is the sheafification of a subpresheaf of $\mathscr{F}$, so can be identified as a subsheaf of $\mathscr{F}$ (similar to what we did earlier with im of morphism).

Combining with earlier natural map (recall $\mathscr{F}|_Z := i^{-1}\mathscr{F}$), we get a sequence $0 \to j_!\mathscr{F}|_U \to \mathscr{F} \to i_*\mathscr{F}|_Z \to 0$. The corresponding sequence on stalks is either $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$ (for $P \in U$) or $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ (for $P \in Z$), so always short exact, i.e. the sequence

$$0 \to j_!\mathscr{F}|_U \to \mathscr{F} \to i_*\mathscr{F}|_Z \to 0$$

is short exact.

Flasque sheaves

We now consider one of the 'simplest' yet most important types of sheaf, whose importance will become clear when we consider cohomology of sheaves.

In general, we cannot of course recover global sections from local sections. But if we can, then clearly the sheaf becomes very simple (say from the perspective of cohomology). The sheaf is called *flasque*, or *flabby*, because it contains 'too much data than necessary'.

Definition (Flasque/flabby sheaves; Hartshorne Exercise 1.16)

A sheaf $\mathscr{F}$ is called **flasque** if every restriction $\mathscr{F}(U) \to \mathscr{F}(V)$ (for $V \subseteq U$) is surjective.

Flasque sheaves

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Example (Constant sheaf of rational functions is flasque)

Given X (irreducible) variety, the constant sheaf on X associated to $k(X)$, equivalently the sheaf of rational functions, is flasque.

Flasque sheaves

A mini-example of the importance of flasque sheaves from the homological viewpoint is the following: every sheaf can be identified naturally as a subsheaf of a flasque sheaf.

Lemma (cf. Hartshorne Exercise 1.16)

Every sheaf $\mathscr{F}$ can be identified naturally as a subsheaf of a flasque sheaf $\mathscr{G}$.

Proof.

For each open U let $\mathscr{G}(U)$ be the functions on U which send each P to a germ in $\mathscr{F}_P$, but with no other restriction. This sheaf is, for good reason, called the *sheaf of discontinuous sections* of $\mathscr{F}$. □

Flasque sheaves

For time reasons we omit the verification of some basic properties of flasque sheaves, which are largely routine but appeal sometimes to Zorn's lemma. E.g. previously we remarked the left exactness of 'taking sections' functor; when $\mathscr{F}'$ is flasque, we can extend it to exactness on the right.

We just remark:

Remark (Importance of flasque sheaves)

One can show that injective $\implies$ flasque (in the usual sense of injective in the abelian category of sheaves) and the previous lemma also shows we can define *flasque resolutions* of sheaves in some canonical sense. We have already noted flasque sheaves are 'simple' in some sense, precisely we have flasque $\implies$ acyclic. This can be one important way to approach cohomology of sheaves.

Glueing sheaves

Before going back full circle to illustrate some application of sheaves to varieties, we first end with a notion which will prove important later when glueing schemes: how do we glue sheaves in the first place?

Proposition (Glueing sheaves; Hartshorne Exercise 1.22)

For X a topo. space and $\{U_i\}$ an open cover, suppose we have sheaves $\mathscr{F}_i$ on U_i which are compatible in the following sense:

- *There are isomorphisms $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ such that $\phi_{ii} = id$ and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$ (cocycle condition)*

Then there is a unique sheaf $\mathscr{F}$ on X formed by glueing the $\mathscr{F}_i$, i.e. there are isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

Remark

It is even possible to glue sheaves which are *a priori* not living in the same topo. space, by first glueing the topo. spaces together, but that would be too much of a topological digression.

Glueing sheaves

Proof.

Again, we focus on the idea and omit the routine verifications.
The idea is that we are clearly able to (and indeed forced to uniquely) define $\mathscr{F}(V)$ on open V, *which are contained in some U_i*, in the obvious way.
Clearly this does not exhaust all opens but crucially **all such open V form a basis of the topology on X.**
So all we need to do is define for each open U

$$\mathscr{F}(U) = \varprojlim_{V \subseteq U, V \text{ basis open set}} \mathscr{F}(V),$$

here we are taking inverse limit. This is also all we can do (so uniqueness) since global sections are determined uniquely by local sections.
The cocycle condition is needed to check that the restrictions $\mathscr{F}|_{U_i}$ are indeed well-defined isomorphic to $\mathscr{F}_i$, because in taking the inverse limit we inherited restriction maps from the $\mathscr{F}_i$. □

Sheaves on basis

From the proof, we realise that we can expect the following: to recover a sheaf, all we need is to know its sections on basis open sets.

Definition (Sheaf on basis)

Given a space X and a basis $\mathcal{B}$, we can define a (pre)sheaf on $\mathcal{B}$ in exactly the same way as a usual sheaf on X, just replacing all "open sets" in the restriction map and glueing axioms with "basis open sets" as appropriate.

Sheaves on basis

Proposition ("Sheafification" of sheaf on basis)

Given a sheaf $\mathscr{F}$ on basis $\mathcal{B}$, there is a unique up to unique isomorphism sheaf $\mathscr{F}'$ on X which extends $\mathscr{F}$.

Proof.

For each open $U \subseteq X$, we let $\mathscr{F}'(U)$ be the functions on U locally defined *on basis open sets* by local sections. Everything proceeds as in our previous sheafification.

Remark that we can verify that this coincides with our previous definition

$$\mathscr{F}'(U) = \varprojlim_{V \subseteq U, V \text{ basis open set}} \mathscr{F}(V),$$

□

Sheaves on basis

Remark (Structure sheaf again)

As mentioned earlier, this allows us to define the structure sheaf $\mathscr{O}_{\text{Spec } A}$ by specifying $\mathscr{O}_{\text{Spec } A}(U_f) = A_f$ for the basis open U_f.

Sheaves on varieties

We now return to the motivating example at the start, i.e. varieties.

Example (Structure sheaf of closed subvariety, sheaf of ideals; Hartshorne Exercise 1.21)

- Let X be a variety with structure sheaf $\mathscr{O}_X$, let Y be a closed subvariety with structure sheaf $\mathscr{O}_Y$, with $\iota : Y \to X$ the inclusion.

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- We may naturally view $\mathscr{O}_Y$ as a sheaf on X via $\iota_*\mathscr{O}_Y$ (cf. also Hartshorne Exercise 1.19 on extending a sheaf by zero).

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- We may naturally view $\mathscr{O}_Y$ as a sheaf on X via $\iota_* \mathscr{O}_Y$ (cf. also Hartshorne Exercise 1.19 on extending a sheaf by zero).
- We have a natural morphism of sheaves $\phi : \mathscr{O}_X \to \iota_* \mathscr{O}_Y$ sending each section f over each open $U \subseteq X$ via $f \mapsto f|_{U \cap Y}$. Actually this is nothing but the pullback of f along ι.

Sheaves on varieties

Example (Structure sheaf of closed subvariety, sheaf of ideals; Hartshorne Exercise 1.21)

- Now denote $\mathscr{I}_Y := \ker \phi$ (recall this is a sheaf). For each open $U \subseteq X$, $\mathscr{I}_Y(U)$ is clearly the ideal of $\mathscr{O}_X(U)$ of regular functions which are zero on $U \cap Y$.

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- $\mathscr{I}_Y$ is called the *sheaf of ideals* of Y, and we have $\mathscr{O}_X / \mathscr{I}_Y \cong \iota_* \mathscr{O}_Y$.

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- $\mathscr{I}_Y$ is called the *sheaf of ideals* of Y, and we have $\mathscr{O}_X / \mathscr{I}_Y \cong \iota_* \mathscr{O}_Y$.
- Observe in particular, $\mathscr{I}_Y(X)$ is the ideal of regular functions on X which are zero on Y. In the affine case, at the level of global sections this coincides with our early notion.

- Next talk: will see how sheaves play a natural central role in (affine) schemes (and indeed in all of the following).

Thank you!