Symmetric spaces and their classification

Bryan Wang Peng Jun

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To understand how M looks like around p, we may head off in any direction $v \in T_pM$ by following the geodesic (think: exponential map).

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Suppose now we stand at a point p of a Riemannian manifold M.

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What is a natural (choice-free, assumption-free) notion of symmetry at p?

Answer: If we head off in the *opposite direction* $-v \in T_pM$, then the manifold should 'look the same'.

Definition

A Riemannian manifold M is **locally symmetric** if around all $p \in M$, there is a local isometry s_p whose differential at p is $-\mathrm{id}$.
Such a local isometry necessarily reflects the geodesics through p.

Definition

M is **(globally) symmetric** (called a **symmetric space**) if the local isometries s_p may all be taken to be global isometries, i.e. extended to the whole M.

(In this talk, all our Riemannian manifolds are WLOG connected.)

A sea of examples...

Example

- Euclidean space $\mathbb{R}^n$ and sphere S^n with their usual metrics and the obvious reflection about each point.
- Projective space $\mathbb{RP}^n$, as $S^n/\{\pm 1\}$.
- Compact Lie group G which must have a bi-invariant metric, with the reflection $g \mapsto g^{-1}$.
- Hyperbolic space $\mathbb{RH}^n$: in the hyperboloid model, one can check the reflection about any point p (w.r.t. the Lorentzian inner product on $\mathbb{R}^{n+1}$) can be taken as s_p.
- The Grassmannian $G(k, n)$ of k-dim subspaces of $\mathbb{R}^n$: for each k-dim subspace E the reflection s_E acts as 1 on E and -1 on $E^\perp$.

Natural question: how does local symmetry relate to **curvature**?

Theorem (Cartan)

M is locally symmetric $\iff$ it has parallel curvature tensor, i.e. $\nabla R = 0$.

Proof sketch

($\Rightarrow$): If M is locally symmetric, then

$$ds_p((\nabla_X R)(Y, Z, W)) = (\nabla_{ds_p X} R)(ds_p Y, ds_p Z, ds_p W)$$

which since $ds_p = -\mathrm{id}$ implies $(\nabla_X R)(Y, Z, W) = 0$.

($\Leftarrow$): This is harder. Idea: exp is a local diffeomorphism, so we can transfer the map $-\mathrm{id}$ on $T_p M$ to M, and suffices to show this is an isometry. For this it suffices to show the curvature remains the same when we head off in both directions, and for this we use radial coordinates with $\nabla_{\partial_r} R = 0$.

We see that locally symmetric spaces can be viewed as a generalisation of space forms.

From local to global

Next natural question: when is a locally symmetric space globally symmetric?

The key point is the extension of the local isometries s_p to global isometries, and the key idea is we can extend them along geodesics.

So the obstructions are:

- Enough geodesics to extend the isometries along: M should be geodesically complete (by Hopf-Rinow theorem, $\iff$ metrically complete)
- Independence of path used to extend the isometries: M should be simply connected.

Theorem (Cartan-Ambrose)

M is locally symmetric + complete + simply-connected $\implies$ M is globally symmetric.

Remark

One can view the preceding results of Cartan and Ambrose as applications of the Cartan-Ambrose-Hicks theorem, which gives existence of local/global isometries from assumptions about curvature.

In the same vein one obtains the classification of space forms which are complete + simply-connected $(S^n, \mathbb{R}^n, \mathbb{RH}^n)$.

These are all in the same circle of ideas.

Aim: classify (globally) symmetric spaces M.

First properties of symmetric spaces:

Lemma

If M is a symmetric space, then

1. *M complete.*
2. *M homogeneous.*

Proof sketch

1. Use the reflections s_p (which reflect geodesics) to extend geodesics indefinitely.
2. For any two points p, q, connect them by a geodesic, and take the reflection about the midpoint s_m.

Given any symmetric space M, consider its universal cover $\tilde{M}$.

$\tilde{M}$ is locally isometric to M, hence locally symmetric; it is also complete; it is simply-connected. Hence we know that $\tilde{M}$ is globally symmetric.

In fact, conversely:

Proposition

M symmetric $\iff$ $M = \tilde{M}/\Gamma$ with Γ discrete and centralising the group of isometries of $\tilde{M}$ generated by transvections (in general, this will just be the identity component of the group of isometries).

Proof sketch

The proof requires the theory of the simply-connected $\tilde{M}$.

In other words, the classification of symmetric spaces is essentially reduced to the **simply-connected case**.

Let M be a simply-connected symmetric space. We saw that M is homogeneous. Fix a point $p \in M$.

Proposition

$G := Iso(M)$ is a Lie group (Myers-Steenrod theorem);
$K := Iso_p(M)$, the isotropy/stabiliser group at p, is compact;
$M \cong G/K$.

We will now try to describe M as much as possible in terms of G.

There are two pieces of data:

1. the symmetric property;
2. the Riemannian structure.

1. The symmetric property:

s_p is an element of G, and it is an involution. We get the involutions:

$$\mathrm{Ad}_{s_p} =: \sigma : G \to G$$
$$\mathrm{Ad}_{s_p} =: s : \mathfrak{g} \to \mathfrak{g}$$

Proposition

$(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$. *In particular, if* $\mathfrak{k}$ *is the Lie algebra of* K, *then* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, *with* $\mathfrak{k}, \mathfrak{p}$ *the* $1, -1$*-eigenspaces of* s *on* $\mathfrak{g}$ *respectively.*

Proof sketch

For any $k \in K$, k and $\sigma(k) = s_p k s_p^{-1}$ have equal differentials at p, hence $k = \sigma(k)$, so $K \subseteq G^\sigma$.
If $g \in G^\sigma$, then by definition it commutes with s_p and hence preserves the fixed-point set of s_p. So if $g \in (G^\sigma)^\circ$ can be connected to the identity $e \in G$, then g must fix p, i.e. $g \in K$, so $(G^\sigma)^\circ \subseteq K$.

Upshot: do not have to work with K, but just an **involution σ of G, or s of $\mathfrak{g}$**. (K is just the fixed points of this involution.)

2 The Riemannian structure.

With $M \cong G/K$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we may view $\mathfrak{p} \cong T_pM$.

K acts faithfully on T_pM (by the differential at p)

$\implies$ $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$ ($[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$).

The Riemannian metric g on T_pM is K-invariant

$\implies$ gives an **inner product B on $\mathfrak{p}$, which is $\mathrm{ad}_{\mathfrak{k}}$-invariant** .

To summarise:

Proposition

Given a symmetric space $M \cong G/K$, we obtain the data $(\mathfrak{g}, s, B)$:

- *A real Lie algebra $\mathfrak{g}$;*
- *An involution s of $\mathfrak{g}$ with ± 1-eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$;*
- *An $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product B on $\mathfrak{p}$.*

Remark

We may also view $\mathfrak{g}$ as the Lie algebra of Killing fields on M; $\mathfrak{k}$ the Killing fields vanishing at p; $\mathfrak{p}$ the infinitesimal transvections, which are the differentials of the one-parameter-subgroups of transvections (translations along geodesics).

Of course, the main point is that the data $(\mathfrak{g}, s, B)$ is sufficient to determine the simply-connected symmetric space M.

Proposition

Given $(\mathfrak{g}, s, B)$, we obtain a simply-connected symmetric space M. Hence there is a correspondence between simply-connected symmetric spaces M and the data $(\mathfrak{g}, s, B)$.

Proof sketch

Let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$, σ an involution of G obtained from s, K the connected Lie subgroup with Lie algebra $\mathfrak{k}$.
Then take $M = G/K$; the inner product B gives the Riemannian structure on M, and σ makes M a symmetric space (for a homogeneous space, having one reflection s_p is enough, since we may take $s_{g(p)} = g \circ s_p \circ g^{-1}$).

Towards classification

Now we want to classify $(\mathfrak{g}, s, B)$.

The crowning achievement of Lie theory is the complete classification of *complex semisimple* Lie algebras $\mathfrak{g}$ (by Cartan and others).

Definition

$\mathfrak{g}$ semisimple $\iff$ its Killing form is non-degenerate $\iff$ it is a direct sum of simple (no proper ideals) Lie algebras

Theorem

All the complex simple Lie algebras are:

- $\mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$
- *one of finitely many low-dimensional exceptions.*

The complex case is simpler as $\mathbb{C}$ is algebraically closed, in particular there is no notion of 'positive-definite', 'negative-definite', etc.

For *real* semisimple Lie algebras $\mathfrak{g}$, one first considers $\mathfrak{g} \otimes \mathbb{C}$ which is complex semisimple, and $\mathfrak{g}$ is called a *real form* of $\mathfrak{g} \otimes \mathbb{C}$. One can then classify the *real forms* of the complex semisimple Lie algebras.

Proposition

Every complex semisimple Lie algebra has a unique compact real form (Lie algebra of a compact Lie group), which is also the unique real form with negative definite Killing form.
The other real forms are called noncompact.

Every real semisimple Lie algebra $\mathfrak{g}$ has a unique (up to conjugation) Cartan involution θ, with corresponding Cartan decomposition into ± 1-eigenspaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
The Killing form is negative-definite on $\mathfrak{k}$, positive-definite on $\mathfrak{p}$, and $\mathfrak{k}, \mathfrak{p}$ are orthogonal.
For the compact real forms, the Cartan involution is trivial, $\mathfrak{g} = \mathfrak{k}$.

Theorem

All the real simple Lie algebras are:

- *compact simple real forms:* $\mathfrak{su}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$
- *noncompact simple real forms:* $\mathfrak{sl}(n, \mathbb{R}), \mathfrak{su}(p, q), \mathfrak{su}^*(2n), \mathfrak{so}(p, q), \mathfrak{so}^*(2n), \mathfrak{sp}(2n, \mathbb{R}), \mathfrak{sp}(p+q)$
- *(noncompact) complex simple Lie algebra viewed as real*
- *finitely many low-dimensional exceptions.*

We want to classify $(\mathfrak{g}, s, B)$. For arbitrary real Lie algebra $\mathfrak{g}$, there is no good classification.

However, it comes with an inner product B!

This is reminiscent of semisimple $\iff$ non-degenerate Killing form, and we should try to compare the two.

Theorem

Every $(\mathfrak{g}, s, B)$ ($\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$) can be decomposed into a direct sum of $(\mathfrak{g}_i, s_i, B_i)$ ($\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$), each of which is of one of 3 types:

- *(Euclidean) $[\mathfrak{p}_i, \mathfrak{p}_i] = 0$.*
- *(Compact irreducible) $\mathfrak{g}_i$ semisimple, $\mathfrak{k}_i$ acts irreducibly on $\mathfrak{p}_i$, the Killing form $\kappa_{\mathfrak{g}_i}$ is negative-definite on $\mathfrak{p}_i$, and B_i is a -ve multiple of $\kappa_{\mathfrak{g}_i}$.*
- *(Noncompact irreducible) $\mathfrak{g}_i$ semisimple, $\mathfrak{k}_i$ acts irreducibly on $\mathfrak{p}_i$, the Killing form $\kappa_{\mathfrak{g}_i}$ is positive-definite on $\mathfrak{p}_i$, and B_i is a $+$ve multiple of $\kappa_{\mathfrak{g}_i}$.*

Proof sketch

Key ideas:

First study the Killing form $\kappa_{\mathfrak{g}}$:
$\mathfrak{k}, \mathfrak{p}$ are orthogonal, and negative-definite on $\mathfrak{k}$.

Next compare B and κ_g on $\mathfrak{p}$: suppose they are represented by matrices $[B], [\kappa_{\mathfrak{g}}]$, then we get the symmetric matrix $A = [\kappa_g][B]^{-1}$ representing an endomorphism of $\mathfrak{p}$.
A has all real eigenvalues. Take the A-eigenspace decomposition of $\mathfrak{p}$ (and then further decompose into irreducible subspaces for the action of $\mathfrak{k}$ on $\mathfrak{p}$).
The zero-eigenspace corresponds to Euclidean type;
the negative eigenspaces corresponds to compact irreducible type;
the positive eigenspaces corresponds to noncompact irreducible type.

The facts about $\kappa_{\mathfrak{g}}$ at the start show non-degeneracy of Killing form, hence semisimplicity of $\mathfrak{g}_i$, for the compact and noncompact types.

Corollary

For a symmetric space M with data $(\mathfrak{g}, s, B)$, we obtain a corresponding decomposition into a product of irreducible symmetric spaces $M = \times_i M_i$.

Remark

The Euclidean type ($[\mathfrak{p}_i, \mathfrak{p}_i] = 0$) corresponds to Euclidean space (with canonical metric).
Apart from this factor of Euclidean space, we are reduced to the $\mathfrak{g}$ semisimple and $(\mathfrak{g}, s, B)$ irreducible setting.

Remark

The requirement that $\mathfrak{k}_i$ act irreducibly on $\mathfrak{p}_i$ is reminiscent of the *de Rham decomposition theorem*, in which a manifold is similarly decomposed according to the irreducible subspaces of the action of holonomy on the tangent space. This is another less direct but more geometric way to understand such a decomposition.

One of the most striking features of the theory is a **duality** between the compact and noncompact types.

Definition (Cartan duality)

Given $(\mathfrak{g}, s, B)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,
the dual $(\mathfrak{g}^*, s^*, B^*)$ is defined by

- $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g} \otimes \mathbb{C}$,
- s^* the obvious restriction of $s \otimes \mathbb{C}$,
- $B^*(iX, iY) = B(X, Y)$ for $X, Y \in \mathfrak{p}$.

One can check that this interchanges compact and noncompact irreducible types.

Finally, the fact that we are in $\mathfrak{g}$ semisimple + Cartan duality allows us to obtain:

Theorem (Classification of irred. $(\mathfrak{g}, s, B)$)

All the irreducible $(\mathfrak{g}, s, B)$ are listed below.
Type 1 is dual to type 3, type 2 is dual to type 4.

Type	$\mathfrak{g}$	s	B
1 (compact)	*compact simple real form*	*(determined by duality from type 3)*	*-ve multiple of $\kappa_{\mathfrak{g}}$*
2 (compact)	*$h \oplus \mathfrak{h}$ ($\mathfrak{h}$ compact simple real form)*	$(X, Y) \mapsto (Y, X)$	*-ve multiple of $\kappa_{\mathfrak{g}}$*
3 (noncompact)	*noncompact simple real form*	*unique Cartan involution θ*	*+ve multiple of $\kappa_{\mathfrak{g}}$*
4 (noncompact)	*complex simple Lie algebra viewed as real*	*(determined by duality from type 2) complex conjugation over compact real form*	*+ve multiple of $\kappa_{\mathfrak{g}}$*

Corollary

The classification of simply-connected symmetric spaces is equivalent to the classification of (compact and noncompact) real forms of Lie algebras!

Theorem (Classification of irred. simply-connected symmetric spaces)

Type	$\mathfrak{g}$	$\mathfrak{k}$	$M = G/K$
1 (compact)	*compact simple real form*	*(TBD next slide)*	*compact homogeneous space (TBD next slide)*
2 (compact)	$\mathfrak{h} \oplus \mathfrak{h}$ *($\mathfrak{h}$ compact simple real form)*	$\mathfrak{h}^{\Delta}$	*compact Lie group $H = SU(n), SO(n), Sp(2n)$ with bi-invariant metric*
3 (noncompact)	*noncompact simple real form*	*unique compact $\mathfrak{k}$ in Cartan decomposition (TBD next slide)*	*noncompact homogeneous space (TBD next slide)*
4 (noncompact)	*complex simple Lie alg. viewed as real*	*compact real form*	$H^{\mathbb{C}}/H = SL(n,\mathbb{C})/SU(n)$, $SO(n,\mathbb{C})/SO(n)$, $Sp(2n,\mathbb{C})/Sp(2n)$

Types 2,4 correspond to compact Lie group (with bi-invariant metric).

Recall there is a classification of all noncompact real forms, and the Cartan involution/decomposition is unique. This allows us to completely classify types 1,3.

Theorem (Classif. of irred. simply-connected symmetric spaces (cont.))

Type 1 (compact)	*Type 3 (noncompact)*
$SU(n)/SO(n)$	$SL(n,\mathbb{R})/SO(n)$
$SU(p+q)/S(U(p)\times U(q))$	$SU(p,q)/S(U(p)\times U(q))$
$SU(2n)/Sp(2n)$	$SU^*(2n)/Sp(2n)$
$SO(p+q)/SO(p)\times SO(q)$	$SO(p,q)/SO(p)\times SO(q)$
$SO(2n)/U(n)$	$SO^*(2n)/U(n)$
$Sp(2n)/U(n)$	$Sp(2n,\mathbb{R})/U(n)$
$Sp(p+q)/Sp(p)\times Sp(q)$	$Sp(p,q)/Sp(p)\times Sp(q)$

and finitely many exceptional ones of 'low' dimension.

Example

Type 1 (compact)	Type 3 (noncompact)
$SO(n+1)/SO(n) \times SO(1)$: sphere S^n	$SO(n,1)/SO(n) \times SO(1)$: hyperbolic space $\mathbb{RH}^n$
$SO(p+q)/SO(p) \times SO(q)$: real Grassmannian	$SO(p,q)/SO(p) \times SO(q)$: hyperbolic Grassmannian
$SU(p+q)/S(U(p) \times U(q))$: complex Grassmannian	$SU(p,q)/S(U(p) \times U(q))$: complex hyperbolic Grassmannian

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In general one sees that the Cartan duality tends to interchange 'usual' spaces and hyperbolic spaces.

Observe that sphere has constant positive sectional curvature while hyperbolic space has constant negative sectional curvature, etc.

This tends to suggest the compact/noncompact classification/Cartan duality may have a deeper geometric meaning.

Theorem

If M is compact/noncompact/Euclidean type, then its sectional curvature is everywhere $\geq 0/\leq 0/= 0$.
Cartan duality flips the sign of the sectional curvature.

Proof sketch

Idea: show by a (technical) computation that $R(X,Y)Z = -[[X,Y],Z]$ and hence the sectional curvature for X, Y orthonormal is, with $B = \lambda\kappa_{\mathfrak{g}}$,

$$\sec(X,Y) = \lambda\kappa_{\mathfrak{g}}([X,Y],[X,Y]);$$

now $[X,Y] \in \mathfrak{k}$ and use negative-definiteness of $\kappa_{\mathfrak{g}}$ on $\mathfrak{k}$.

Theorem

If M is compact/noncompact irreducible (or Euclidean), then M Einstein with Einstein constant $> 0/ < 0$ (or 0).

Proof sketch

The Ricci (1,1)-tensor and all its eigenspaces are K-invariant, so M being irreducible means there can be only one eigenspace, i.e. M is Einstein.

Thank you!