TQFT seminar notes for presentation on 1 Mar

2D TQFTs and Frobenius algebras

simp. non-triv. eg. of TQFT fr. alg. perspective & simp. non-triv. eg. of TQFT fr. phys.'s perspective

understand what is going on in TQFT w/o analytic difficulties

finite gp. version of Chern-Simons (Dijkgraaf-Witten theory) of path integral

Today's aim: 'Classify' all 2D TQFTs and understand an important first example of TQFT.

come together in a very nice way!

Recall last week: definition of TQFT + classificat^{n} of 1-D TQFT.
↳ completely uninteresting!

Let's review definition of TQFT as certain things will become impt.

~~Cate~~ Def of $(n-1)+1$-dim TQFT: (today, $n=2$).

Category Bord_n: Objects: closed, oriented $(n-1)$-folds / (orientat^{n}-preserving) diffeomorphisms

Morphisms: (oriented) bordisms $\overset{M}{\wedge}$ fr. E to F / (orientat^{n}-pres.) diffeomorphism rel. E, F
i.e. "hold E, F completely fixed"

Convention: $\underset{E}{\longrightarrow}\underset{F}{}$. Morphisms $E \to F$

$\}\boxed{M}\{$

Bord_n has monoidal structure: obvious disjoint union.

+ symmetric (monoidal) structure:

recall in $\mathrm{Vect}_{\mathbb{F}}$, symm. structure given by $U \otimes V \xrightarrow{\sim} V \otimes U$
$u \otimes v \longmapsto v \otimes u$.
"braiding".

In Bord_n,

: NOT (in general) identity even if $U = V$!
(recall equiv. of morphisms must hold $U \sqcup V$ and $V \sqcup U$ fixed).

"braiding"

TQFT is symm. monoidal functor $f: \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\mathbb{F}}$.

sends braiding to braiding!

sends ~~braiding~~ $\sqcup$ to $\otimes$.

1-D
↳ objects: pts $\cdot_{+}$ $\cdot_{-}$ } two different objects.

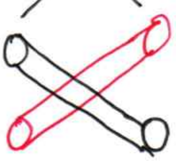
2-D ~ (orientat^{n}-preserving) diffeomorphic to $\bigcirc$. So ~~general~~ "all" objects are $\phi, \bigcirc, \bigcirc\bigcirc, \ldots$

Upshot: since we identify $\bigcirc$ and $\bigcirc$, there is no need to keep track of orientat^{n}'s today (or rather, all orientat^{n}'s are implicit).

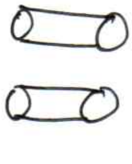
Morphisms:

identified as same object.

↳ E.g.



Is this $\simeq$



? Ans: No! As we have to hold source + target fixed: the colour coding makes this clear (cannot send black to red)

However: TQFT being a symm. functor means it must always send

braiding

to $(u \otimes v \longmapsto v \otimes u)$ braiding. Hence why we usually do not think much of it.

Recall Aim 1: classify 2D TQFTs. So let Z denote any 2D TQFT.

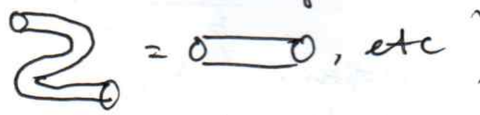
Since we focused on duality last week, today we will take an approach that emphasizes that.

In particular recall last week: because we had a morphism that looks like this: , we saw:

① $Z(0)$ is f.d., say $= V$.

② $Z(0)$ is its dual $\rightsquigarrow$ what we mean is $Z(\;)$ gives a non-degenerate pairing $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$.

③ In fact, from the proof last week (recall we had things like



we can deduce $Z(\;)$ must be the coevaluation map of the pairing $\langle\cdot,\cdot\rangle$; i.e. it sends $1 \longmapsto \sum v_i \otimes v_i^*$.

In 1-D case, by considering ~~Z~~ , it is essentially the end of the story.

Q: what can we do in 2-D that we can't in 1-D?

a) Recall last week we also saw this interesting feature of TQFT: if we have a cobordism , then $Z(\;)$ $: \mathbb{F} \to V$ essentially picks out an element of V. (what is this ele. of V?)

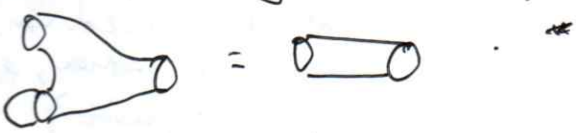
(Can't do this in 1-D!)

b) Prototypical example when we say cobordism:

$Z(\;)$ is a map $V \otimes V \to V$ (Can't do this in 1-D!)

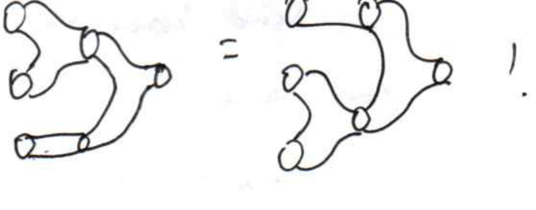
This is nothing but a multiplication in V, making V an $\mathbb{F}$-algebra!

check: - unit? Is precisely the ele given by , because of



"$v \otimes 1 \longmapsto v$".

- associativity? Yes, as



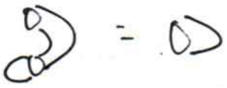
So now: given 2D TQFT Z, get V a f.d. $\mathbb{F}$-alg (unital, associative)

Non-deg pairg can come f. anywhere.

with a non-deg. pairing $\langle \cdot, \cdot \rangle$.

Is that all? None ... and ... look similar. Let's try to apply the maneuvers of unital & associativity to ... .

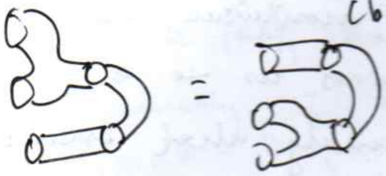
- Unit:



, a linear functional $V \to \mathbb{F}$ defined by $v \mapsto \langle v, 1 \rangle$.

Nothing much, ~~but this will~~ for now (but we will return to it later!)

- Associativity:



i.e. $\langle xy, z \rangle = \langle x, yz \rangle$.

Such $\langle \cdot, \cdot \rangle$ is also called associative!

Upshot: given 2D TQFT Z, $\boxed{\text{get } V \text{ a f.d. } \mathbb{F}\text{-alg (unital, assoc.) with non-deg associative pairing } \langle \cdot, \cdot \rangle}$

$\|$

Def. V a Frobenius alg.!

One of the most important properties in algebra: commutativity — comm. alg / non-comm. alg. (assoc. algs.)

Is V comm.? Yes!! Check:

"$\begin{matrix} x \\ y \end{matrix} \mapsto xy$" "$\begin{matrix} x \\ y \end{matrix} \mapsto \begin{matrix} y \\ x \end{matrix} \mapsto yx$"

Recall need hold source and target fixed. But we can just twist!

Have shown:

Prop: Given 2-D TQFT Z, get a comm. Frob. alg. V.

In fact:

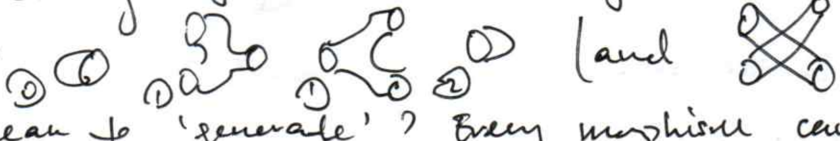
Thm: The converse is true: a comm. Frob. alg V is sufficient to fully define a 2D TQFT Z.

How to prove this? Forget abt having TQFT. Just start w V. First qn: what is needed to define a 2-D TQFT?

Recall for 1-D TQFT: look at generators & relations.

For 2-D TQFT, we can do the same b/c we have classification of surfaces.

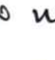
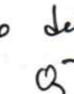
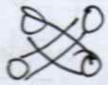
Prop. Bord_2 is generated by object $\bigcirc$ (under disjoint union) and morphisms (and twist) } ele. cobordisms

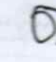


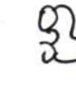
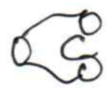
Rmk: What does it mean to 'generate'? Every morphism can be written as a composite/disjoint union where in each piece, just one of the above 'elementary' morphisms.

Pf: Morse theory (indices of critical points corr. to the above).

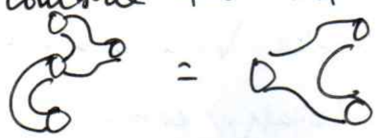
Rmk: Why can't we do the same in 3D and higher? Actually, clue is in the above use of morse theory: a critical pt. of index 1 is just a saddle: how do we know it is locally a pair of pants (we can already see there are two diff. directs, so this is not sth trivial) This requires a detailed argument which is only feasible in the 2D case (cf. Hirsch, Diff. Top) ⇝ ~~e~~essentially the pf of classificat° of surfaces.
This also hints at the need for extended TQFT: if we allow just a saddle by itself to count as a morphism, then such obstacles would be reduced. But that requires us to also allow 0-dim mfds in a 2D TQFT: sth which is precisely what extended TQFT does!

So need to define TQFT on these elementary bordisms. We alr know what  and  have to be: the unit & multiplicat°. So suff. consider  and . (Also we earlier said  is also known.)

Now we have alr seen  but deferred it to later!
It must be the linear f°al given by $v \mapsto \langle v, 1 \rangle$ ($= \langle 1, v \rangle$ by assoc.).
[Digression: There is 1-1 corr. between assoc. pairings and linear functionals.
Given assoc. pairing, we have seen how to produce a linear functional.
Conversely, given linear functional ε , from "  =  " we see that a pairing should be given by $\langle x.y \rangle = \varepsilon(xy)$.
The non-deg condition corr. to: ~~ε~~ ker ε has no (non-trivial) (left) ideals. (Algebra exercise!)
So can also define Frob. alg. by existence of such linear f°al: more useful algebraically, but the ~~non-de~~ condit° has no good interpretat° in TQFT setting.]

More interesting one is . Recall that for  we compared it with . So now we compare it with ; recall at the start we said this must be the ~~copa~~ coevaluation assoc. to the non-deg. pairing , i.e. $1 \mapsto \sum v_i \otimes v_i^*$.
How to define  using  and what we ~~a~~ currently have?

~~Case~~ Start w  : now need one input  and now have three outputs V so should combine two into one: (but want two)



∴ (If we have such a $V \to V \otimes V$ which can be used to define a TQFT, it has to be equal to this composit. ~~§~~ Same for .) & what we are saying.

In other words ~~it~~ it must correspond to $V \to V \otimes V$ sending

$$v \mapsto \sum v \otimes v_i \otimes v_i^* \mapsto \sum (v v_i) \otimes v_i^* .$$

Obvious qu: does this coincide with  ? If it doesn't, then there is no hope of defining a TQFT starting with V. This is sth that has to be checked independently of TQFT considerations, i.e. purely algebraically:

$$\text{Is } \sum (v v_i) \otimes v_i^* \overset{?}{=} \sum v_i \otimes (v_i^* v) \;?$$

Standard technique: have non-deg form $\langle \cdot , \cdot \rangle$, its basis & dual basis.

Multiplication $v v_i$ is hard to handle. So write

$$\sum_{i,j} \langle v v_i, v_j^* \rangle v_j \otimes v_i^* \qquad \sum_{i,j} v_i \otimes v_j^* \langle v_j, v_i^* v \rangle .$$

$$\sum_{i,j} v_i \otimes v_j^* \langle v v_j, v_i^* \rangle .$$

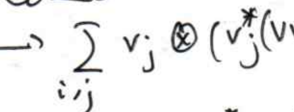
Now swap i,j.

So get a 'canonical' $V \to V \otimes V$ starting f. V a comm. Frob. alg.

This is a comultiplication on V!

Similar to multiplication: check:

(1) Counit: a $V \to \mathbb{F}$, obvious candidate is  !

Want sth like "  "

i.e. $\sum \langle v v_i, 1 \rangle v_i^* = v$? Yes!

(2) Coassociativity: "  "

$$v \mapsto \sum (v v_i) \otimes v_i^* \mapsto \sum_{i,j} v_j \otimes (v_j^* (v v_i)) \otimes v_i$$

$$v \mapsto \sum v_j \otimes (v_j^* v) \mapsto \sum_{i,j} v_j \otimes ((v_j^* v) v_i^*) \otimes v_i$$

} so essentially follows from associativity.

(3) Cocommutativity: "  "

$$v \mapsto \sum (v v_i) \otimes v_i^* \mapsto \sum v_i^* \otimes (v v_i)$$

$$v \mapsto \sum v_i^* \otimes (v_i v)$$

} so essentially follows from commutativity.

Rmk: Now V has unit, mult, comult, counit which ~~will be~~ correspond to the four elementary bordisms to be used to define the 2D TQFT.

Rmk that this does NOT make V a bialgebra, as the comult here sends $1 \mapsto \sum v_i \otimes v_i^*$ (where in bialg. comult sends $1 \mapsto 1 \otimes 1$.)

Although some impt bialg (e.g. Hopf alg) are also then Frob. alg..

this means we have two different comults on the alg.

Now we have specified ~~what the TQ~~ what map the TQFT should send each ele. cobordism to.

Now we need to check the most impt thing: is this well-defined??

i.e. given any ~~two~~ cobordism and two diff. decompositions into ele. cobordisms, do the corr. linear maps agree?

Sketch pf.

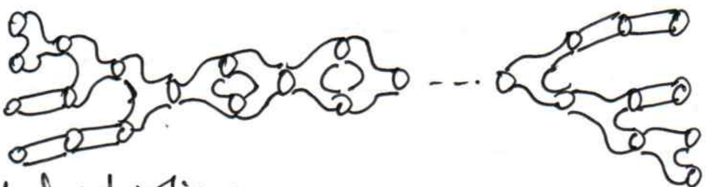
~~Pf~~ idea: Given two decomps, ~~put them~~ show that they both ~~into~~ coincide w the lin. map attached to a certain normal form by relations which we know hold between the lin. maps assigned to the ele. cobordisms:

These two normal forms must coincide by genus coincidents, etc.

Steps: (1) Do above for conn. cobordisms w/o braid  ⇝ this is the meat of the proof!

(2) For conn. w braid , inductively remove them (not immediate but just technicality)

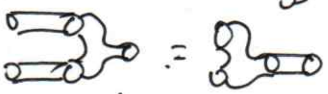
(3) For disconnected cobordisms: just add some ~~tw~~ braidings at the end to view them as disjoint unions (✓).

Normal form: 

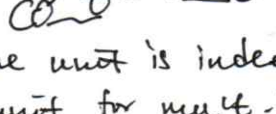
(if source = ϕ, then start w; similar for target).

(1) How to bring connected cobordism to normal form? Move all left (& all right.)

Along the way of moving left, what can meet? (resp. right)

- Identity: pass through. 

(id ∘ mult = mult ∘ id)

- : disappear: 

(in V, the unit is indeed unit for mult.)

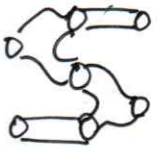
(resp. : disappear )

(counit)

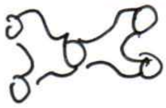
- : two cases.

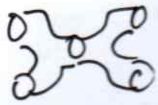
(resp. ...)

a)  : then just view it as middle part of normal form.

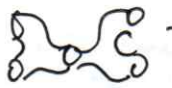
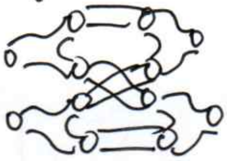
b)  : This represents a composition

$$v \otimes w \longmapsto \sum v_i \otimes (v_i^* v) \otimes w \longmapsto \sum v_i \otimes (v_i^* v w)$$

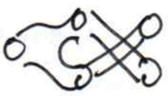
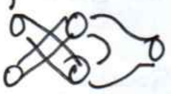
which is hence equal to 

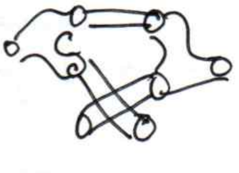
Similarly  = 

This is the most impt relation, called the Frobenius relation, and ∴ we have exhausted all cases, also the last of the generating relations we need.

Rmk: This relation is about "how multiplication & comultiplication commute". In a bialgebra, comult is an alg. hom., so mult & comult commute by  = 

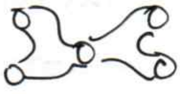
Step ② To handle  : can assume inductively ~~some~~ surrounding* areas are in normal form. So only a few cases:

 (resp. ) : cocomm. resp. comm.

 : $v \otimes w \longmapsto \sum (v v_i) \otimes v_i^* \otimes w$

$\longmapsto \sum (v v_i) \otimes w \otimes v_i^*$

$\longmapsto \sum (v w v_i) \otimes v_i^*$

so it is equal to  again, allowing us to remove this braiding.

This completes the proof (sketch) of the main thm. $\square$

Remarks on pf of the main thm:

Proof is ~~boring!~~ interesting but boring: because it can't generalise!

~~Focus on two key takeaways: more in~~

However, it can help us answer a more interesting qn:

as always, when we want to study sth, we should always start w the simplest case & see what it tells us.

What if we want to generalise this to higher dimensions?

What works:

1-D: specify on $\cdot^+$.

2-D: specify on $\bigcirc$.

~~Morse theory to split into elementary cobords:~~

Duality: 1-D: tells you $\cdot^-$

2-D: non-deg. pairing.

Morse theory to split into elementary cobordisms.

What doesn't:

3-D ~~& above~~ (say): specify on all surfaces of genus g ?? $(g \geq 0)$

Fix: extended TQFT: extend all the way down to 0-folds (pt) and specify on pt!

"(Cobordism hypothesis: suffice to specify on pt!)"

~~Fix:~~ Cobordism hypothesis: use duality data in essential way.

Notion of "fully dualizable"

E.g. in 2-D case: need (much more) extra work to say that saddle pt is just pair of pants.

Fix: extended TQFT to split into more local cobordisms (e.g. just saddle itself). This also gives us more flexibility when using it to compute invariants (next).

In particular: above treatment of TQFTs requires us to understand all surfaces completely. ~~Still~~, So using it to then compute invariants attached to surfaces is somewhat 'reverse':

What we will do next: illustrate how TQFT is usually used to compute invariants: find a TQFT defined 'by nature' i.e. 'globally', then see what it does locally and use that to compute.

After the break....

Example .

(& other similar things e.g. Hopf alg)

Most standard example of Frobenius alg. from algebraist's perspective:

group alg. $\mathbb{C}[G]$ (char 0, alg. closed) for a finite gp G.

of course, this is not comm. Take its center V: those elements with same coefficient on each conjugacy class. $\simeq$ class funct's on G (w product as convolut'n).

Rep theory of finite gps: characters of reps are class f'ns, and there is a bilinear form

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum \phi(g)\psi(g^{-1})$$

wrt which the chars of the irreps $\phi_1, \dots, \phi_k$ of G form an orthonormal basis (b/c this form essentially computes the dim of the Hom space).

Can check this pairing is assoc., so V is ~~naturally~~ a Frob alg. w this pairing.

The unit 1 ~~= char of triv. rep~~.

and coun. lin. f'nal (counit) $\phi \mapsto \langle \phi, 1 \rangle = \frac{\phi(1)}{|G|}$ ~~$\left(= \frac{\dim V_\phi}{|G|}\right)$ where V_ϕ~~

$\left(= \frac{\dim V_\phi}{|G|} \right.$ if ϕ is char of irrep & V_ϕ the corr. rep. $\left.\right)$

~~Since have ONB~~

~~Let the chars of the irreps be $\phi_1, \dots, \phi_k$ (k = # conj. classes of G)~~

~~Last thing we don't know abt V: its comultiplicat'n Δ~~

~~Suppose it sends $\Delta(\phi_i) = \sum c^i_{jk} \phi_j \otimes \phi_k$.~~

~~Then Δ has to satisfy counit: $\phi_i = \sum c^i_{jk} \frac{\phi_j(1)}{|G|} \phi_k = \sum c^i_{jk} \phi_j \frac{\phi_k(1)}{|G|}$~~

Comult is simple: $v \mapsto \sum (v\phi_i) \otimes \phi_i$.

So Need to understand the multiplicat'n and comult in V wrt $\phi_1, \dots, \phi_k$.

In fact one can show $\psi_i = \frac{\phi_i(1)}{|G|} \phi_i$ satisfy $\psi_i^2 = \psi_i$, $\psi_i \psi_j = 0$.

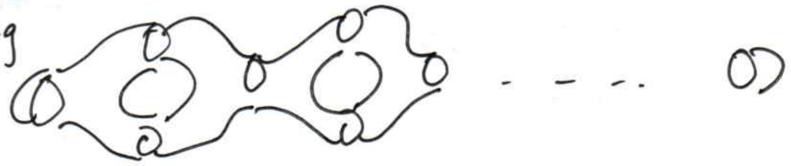
(why? The ψ_i act as scalars on each irrep, taking trace & usg the orthonormality relation we see they act as id on its corr. irrep and 0 on the others, so they are the primitive central idempotents in the block decomp. of $\mathbb{C}[G]$.)

So $\phi_i^2 = \left(\frac{\phi_i(1)}{|G|}\right)^{-1} \phi_i$, $\phi_i \phi_j = 0$.

~~By thm~~ V defines a TQFT Z, and ~~Now we~~ Now want to use Z to compute invariants of surfaces.

Closed surface ν of genus g: compute partit. f. locally in normal form:

Σ_g



$\cancel{\Sigma \mapsto \sum \phi_i \otimes \phi_i \mapsto}$

each of these sends

$\nu \mapsto \sum (\nu\phi_i)\phi_i \mapsto \nu(\sum \phi_i^2)$

$\cancel{\nu(\sum\phi_i)} = \nu(\sum \phi_i^2)$

So

(recall $\phi_i\phi_j = 0$)

$$1 \longmapsto \sum \phi_i^{2g}$$

$$= \sum\left(\frac{\phi_i(1)}{|G|}\right)^{1-2g}\phi_i \mapsto \sum\left(\frac{\phi_i(1)}{|G|}\right)^{2-2g} = \sum\left(\frac{\dim V_i}{|G|}\right)^{\chi(\Sigma_g)}$$

Observe: only invariant of Σ_g we get is just its Euler characteristic.

This whole process hinges on fact that we know classificat. of surfaces;

A ~~does not~~ cannot a priori give us any interesting invariants beyond those which we already know f. classificat. of surfaces (i.e. genus, χ, etc)

Also: above const. is artificial & essentially rep-theoretic.

Does it arise more naturally?

Soln: ~~To have hope of~~ We need to define the TQFT by a process independently of the known classificat. of surfaces. Then we can check that the local pieces agree with the local def. above, and the computat. of the invariant will be by chopping into pieces as above.

Recall from last week the "motivat. for TQFT".

To each object in $Bord_n$ ($(n-1)$-fold) is assigned a Hilbert space which is a space of funct.s on some kind of space of fields.

So here suppose for E a 1-fold we ~~assign~~ have "space of fields = principal G-bundles over E" $Prin_G(E)$

and the corr. "Hilbert space" is funct.s on iso. classes of prin. G-bundles over E. $L^2(\overline{Prin_G(E)})$

To a cobordism should corr. a linear map between the two f. spaces which should be given by a sort of path integral.

So need measure on "space of fields" -

For $P \in \overline{\mathrm{Prin}_G(E)}$, define $\mu(P) = \dfrac{1}{|\mathrm{Aut}(P)|}$. Why? This ~~#~~ gives the so-called groupoid <u>cardinality</u>! not so well-known but lots of interesting theory. cf. Terence Tao's blog post, and in fact what we will do here has been studied systematically by Baez, Hoffnung–Walker as "degroupoidification"

Given cobordism

$E \qquad F$.

Want lin. f'al $L^2(\overline{\mathrm{Prin}_G(E)}) \longrightarrow L^2(\overline{\mathrm{Prin}_G(F)})$.

Have diagram

$$\begin{array}{ccccc} & & \overline{\mathrm{Prin}_G(M)} \ni z & & \\ & \swarrow^{\text{restrict}^{\underline{n}}} & & \searrow^{\text{restrict}^{\underline{n}}} & \\ \overline{\mathrm{Prin}_G(E)} & & & & \overline{\mathrm{Prin}_G(F)} \ni y \end{array}$$

Given ϕ $\xrightarrow{\text{pullback}}$ ϕ^*; ψ: How to pushforward? Integration along fibers.

New function ψ should ~~send~~ ~~have~~ send $y \in \overline{\mathrm{Prin}_G(F)}$

$$\psi(y)\,\mu(y) = \int_{\substack{z \in \overline{\mathrm{Prin}_G(M)} \\ \text{s.t. } z \text{ restricts to } y}} \phi^*(z)\,\mu(z)$$

→ integrating over things with a certain boundary condit$^{\underline{n}}$

"path integral" which is just a finite sum! In fin. gp. case, no analytic difficulties, everything is rigorous.

$$\psi(y) = |\mathrm{Aut}(y)| \sum_{\substack{z \in \overline{\mathrm{Prin}_G(M)} \\ z \text{ restricts to } y}} \phi^*(z)\,\frac{1}{|\mathrm{Aut}(z)|}$$

$$= |\mathrm{Aut}(y)| \sum_{z} \phi(z|_E)\,\frac{1}{|\mathrm{Aut}(z)|}$$

Here the action funct'al is trivial.

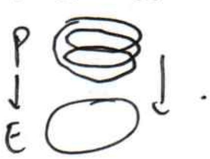
Can check this is indeed functorial (respects composit$^{\underline{n}}$) (can refer to Baez paper above) so this defines a TQFT independent of classificat$^{\underline{n}}$ of surfaces!

Concretely, what does this mean, <u>in our example</u>, esp.?

Need to understand the main player "space of fields" $\overline{\mathrm{Prin}_G(E)}$ better, with $E = S^1$ as our guiding example.

First of all: since G ~~finite~~ discrete, prin. G-bundle is just a (regular) covering space over E, <u>not nec. connected</u>. (have (simply) transitive action of G)

E.g. $E = S^1$, $G = \mathbb{Z}/3\mathbb{Z}$



We know such covering spaces are classified by quotients of $\pi_1(E)$ (its group of deck transformat$^{\underline{n}}$s). $\mathrm{Deck}(p)$.

In this way one can establish:

~~$\mathrm{Prin}_G(E) \simeq \mathrm{Hom}(\pi_1(E), G)$~~.

G connected

Prop: ∧ $\overline{\mathrm{Prin}_G(E)} \simeq \mathrm{Hom}(\pi_1(E), G) / G\text{-conj}$.

If want to look at 'all' $\mathrm{Prin}_G(E)$ i.e. with a fixed basepoint then $\mathrm{Prin}_G(E) \simeq \mathrm{Hom}(\pi_1(E), G)$

Rmks:
- Deck(p) ~~gen fact~~ is a subgp of G, but need not be whole G.
- ~~Need to consider~~ Principal G-bundle has extra structure (G-action), so consider maps $\mathrm{Hom}(\pi_1(E), G)$ (i.e. embeddings of Deck(p) into G) → here, essentially implicitly have a choice of base pt.
- In the language of G-bdles, ~~Explicitly, one~~ the identificat- is via holonomy / monodromy since G discrete there is a natural flat connect- on any such bundle; giving notion of "parallel transport": locally we have ↓ E which allows us to consider holonomy: given an ele. of $\pi_1(E)$, going once around the corr. loop we may end up at a diff. pt., giving a corr. ele. of G (essentially manifestation of the fact it is cov. sp.: given a loop in E, may lift it to P)
- When are two bundles equivalent ~~Isomorphisms of G-bundles~~ will conjugate the holonomy. (Example for $G = \mathbb{Z}/3\mathbb{Z}$) are locally just left action by G, so it will conjugate the embedding.

Principal G-bundles have more structure than covering spaces, so this ~~proof~~ approach is rather low-tech; still a few details to check + make it all precise.

A more high-brow proof: recall $\overline{\mathrm{Prin}_G(E)} \simeq [E, BG]$

In fact for G discrete, using long exact homotopy seq on $EG \to BG$ and fact that characterizing prop. of univ. bundle is EG has all trivial homotopy gps (weakly contractible), we see that BG is a $K(G,1)$.

Then ~~ΣE,~~ for a base pt $e \in E$, $[E, BG]_{\text{base pt}} \simeq \mathrm{Hom}(\pi_1(E), G)$ (CP4 in Chap 1 of Hatcher's alg. top textbook; essentially obstruct- theoretic)

so $[E, BG] \simeq \mathrm{Hom}(\pi_1(E), G) / G\text{-conj}$.

Back to our example.

We see $Z(O)$ = f-s on conjugacy classes of $G = V$

Multiplicat- = $\left.\begin{matrix} f \\ g \end{matrix}\right\} h$ Given two ~~f~~ class f-s f, g,

the corresponding funct- h is obtained by

$$h(y) = |\mathrm{Aut}(y)| \sum_{\substack{z \in \mathrm{Prin}_G(\text{pants}) \\ z \text{ restricts to } y}} (f \otimes g)\left(z\big|_{0}^{0}\right) \frac{1}{|\mathrm{Aut}(z)|}$$

$y \in {}^{G}/_{G\text{-conj}} \cong \mathrm{Prin}_G(O)$

 has $\pi_1 \simeq \mathbb{Z} * \mathbb{Z}$; given prin. bundle, if holonomy around top leg $= y_1$ and around bottom leg $= y_2$ then around out leg $= gh$.

$$\overset{\text{S}'}{=} |C_G(y)| \sum_{\substack{(y_1, y_2) \in G \times G/G\text{-conj.} \\ y_1 y_2 \text{ conj. to } y}} f(y_1)\, g(y_2) \frac{1}{|C_{G\times G}(y_1, y_2)|}$$

$$\overset{\text{orbit-stab.}}{=} |C_G(y)| \sum_{\substack{(y_1, y_2) \in G \times G \\ y_1 y_2 \text{ conj. to } y}} f(y_1)\, g(y_2) \frac{1}{|G|}$$

$$\overset{\text{orbit-stab}}{=} \frac{1}{|\#\text{conj. class of } y|} \sum_{\substack{(y_1, y_2) \in G \times G \\ y_1 y_2 \text{ conj. to } y}} f(y_1)\, g(y_2)$$

$$= \sum_{y_1 y_2 = y} f(y_1)\, g(y_2) \quad (\because f, g \text{ are class f}^{\text{n}}\text{s})$$

which is indeed convolution!

Similarly one checks unit O and  pairing are as we defined them using the Frob. alg. V (they are similar & easier than the multiplicat^{n}).

So this does indeed give the <u>same</u> TQFT Z.

However, what is the partit^{n} f^{n} this time?

$$Z(\text{genus } g \text{ surface } \Sigma_g) = \sum_{z \in \mathrm{Prin}_G(\Sigma_g)} \frac{1}{|\mathrm{Aut}(z)|} \quad \text{(groupoid cardinality)}$$

$$= \sum_{z \in \mathrm{Hom}(\pi_1(\Sigma_g), G)/G\text{-conj.}} \frac{1}{|C_G(z)|}$$

$$\overset{\text{orbit-stab.}}{=} \sum_{z \in \mathrm{Hom}(\pi_1(\Sigma_g), G)} \frac{1}{|G|} = \frac{|\mathrm{Hom}(\pi_1(\Sigma_g), G)|}{|G|}$$

which counts principal G-bundles on Σ_g.

Combin'g w/ prev., we see that we may compute this invariant by

$$\frac{|\mathrm{Hom}(\pi_1(\Sigma_g), G)|}{|G|} = \sum \left(\frac{\dim V_i}{|G|}\right)^{\chi(\Sigma_g)}, \quad \nabla_{\!\!\circ}.$$

For ~~con~~ complicated surfaces i.e. high g, LHS is hard to compute whole RHS is very easy!

This is perhaps the simplest important example of computing an invariant (of closed surfaces) using TQFT.

Next week: go towards 3-dimensional manifolds.