# Knot invariants, tensor categories and quantum groups: why, how and what next

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Suppose: want to study geometric objects.

Geometry is ‘hard’.

Easier: study numerical, algebraic invariants attached to them.

**Philosophy: invariants should be functors**

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$

$$\textbf{‘geometry’} \to \textbf{‘algebra’}$$

Start with **knots** (and **links** = knots with more than one component): simplest geometric object with rich theory and unanswered questions.

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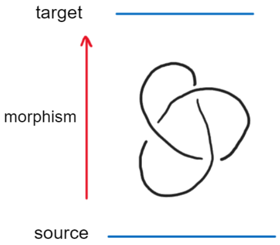
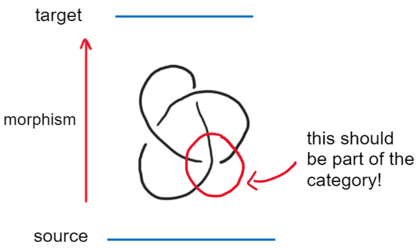
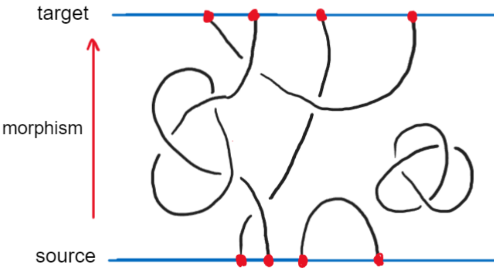



Knots = morphisms from empty object to empty object.




Key point in geometry: want to be able to study things 'locally'.

In this case, have: endpoints.




Objects = endpoints (fixed);
Morphisms = tangles (up to isotopy);
Composition = vertical stacking.

This is the tangle category $\mathscr{T}$.

**Philosophy: invariants should be functors from geometry to algebra.**

Question 2: what algebraic category?

Naturally: the category $\mathscr{C}$ of (f.d.) representations of an algebra $A$.

# The dictionary

Question 3: how to define $\mathscr{F} : \mathscr{T} \to \mathscr{C}$?

We will build this 'dictionary':

| Axiom | $\mathscr{T}$ (tangles) | $\mathscr{C}$ ($A$-mods) | $A$ |
|---|---|---|---|
| Monoidal | Horizontal composition | Tensor product | Bialgebra (comult $\Delta$, counit $\epsilon$) |
| Braided | Braiding | $V_1 \otimes V_2 \cong V_2 \otimes V_1$ | Quasitriangular ($R$-matrix) |
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2. $\mathscr{C}$ and $A$: reconstruction theory (in a heuristic sense).

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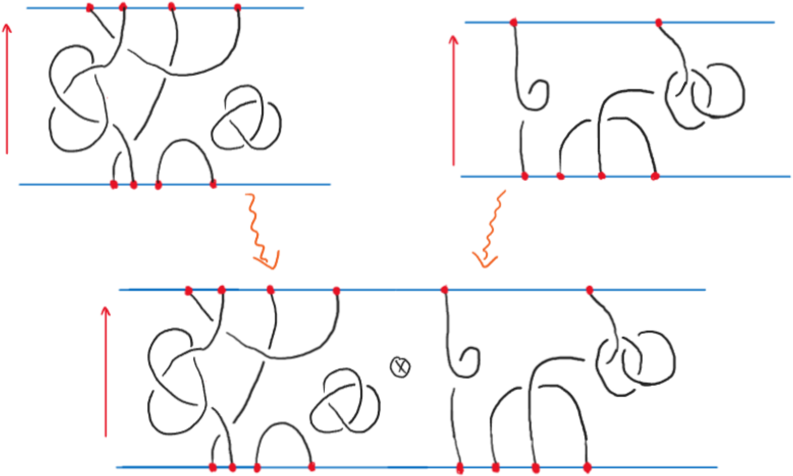
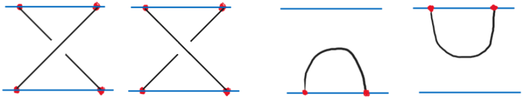
# Monoidal - geometric motivation

Putting two tangles side-by-side (horizontal composition) makes $\mathscr{T}$ a (strict) monoidal category via horizontal composition:

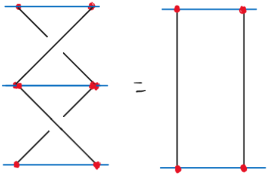


Recall the point of introducing categories was to study the geometry locally.

With isotopy, horizontal and vertical composition, it is intuitive to see that we can chop any tangle up into sufficiently small pieces (generators):



The first two are inverses of each other:



(id)

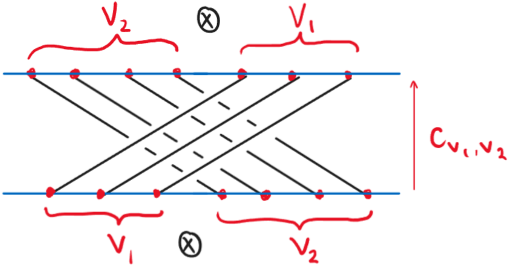
so just need to focus on the first.

# Braiding - geometric motivation

The first is an instance of a general geometric phenomenon:

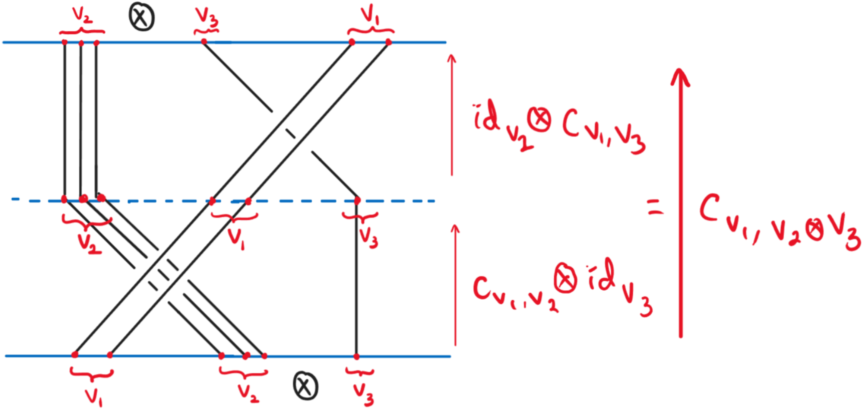
Suppose we have two objects in $\mathscr{T}$:


Then there is a particular 'braiding' tangle:




which is a particular 'braiding' morphism $c_{V_1,V_2}$.

This morphism is compatible with the monoidal structure $\otimes$ in the following way:

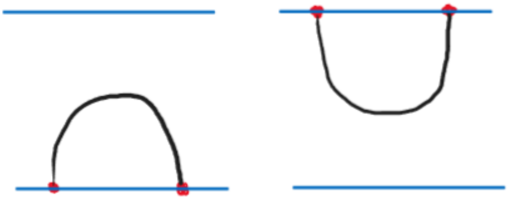



This is precisely the axioms defining a braided (monoidal) category!

If we were only interested in braids (tangles where the strands only go upwards), then we could well stop here.

However, strands can go up then down, or down then up.

This brings us to the second set of generators:

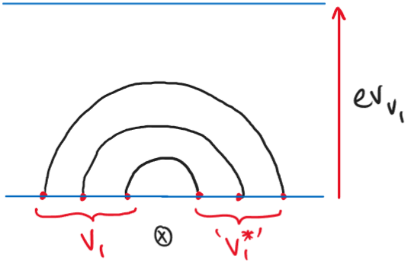


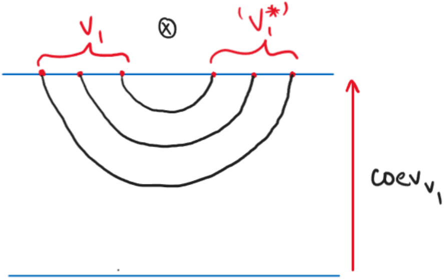
# Duality - geometric motivation

Again, this is an instance of a general geometric phenomenon.

Suppose we have any object:

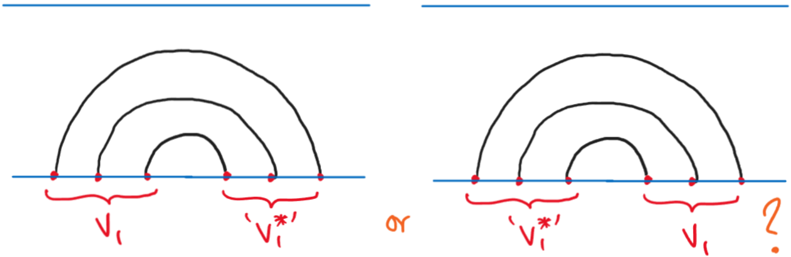

Then there are particular ‘cap’ and ‘cup’ tangles:



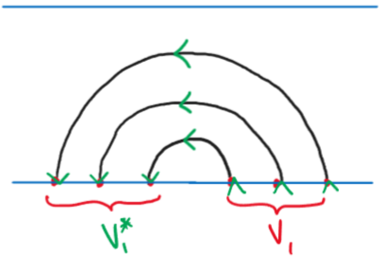

giving a ‘dual’ for $V_1$ with ‘evaluation’ and ‘coevaluation’ maps.

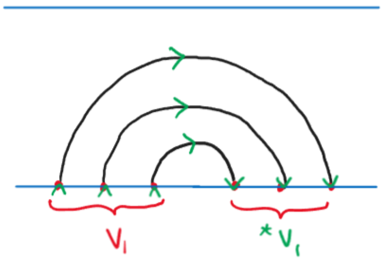
Immediately we see a heuristic problem: there is no way to differentiate between $V_1 \otimes V_1^*$ and $V_1^* \otimes V_1$!



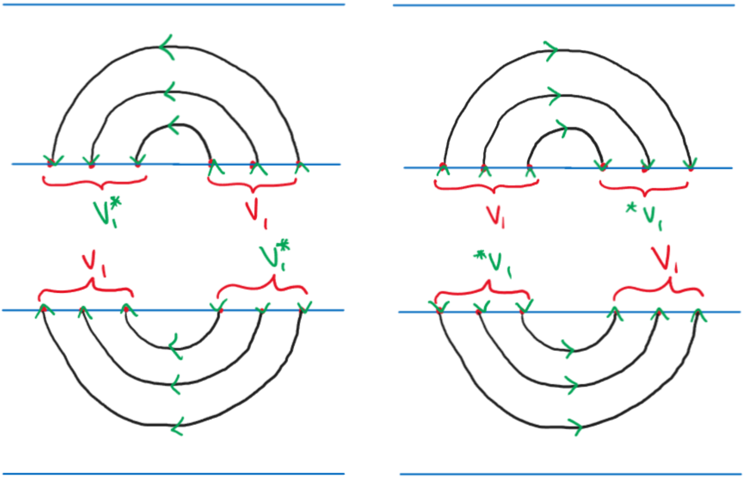
Solution: work with **oriented** tangles (so we study additional geometric structure and this can only strengthen our invariant anyway).

Objects are now points with upward/downward orientation, and strands carry orientation.



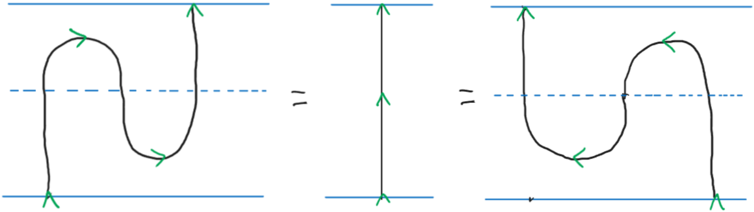



Pointing to the left = left dual; Pointing to the right = right dual!




(assuming $V_1$ is upward pointing)

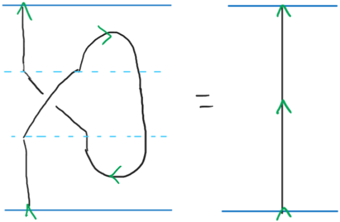
Cup and cap are compatible in the following way:



This is precisely the axioms for duals in a (monoidal) category!

Braiding and duality are independent categorical concepts.

In $\mathscr{T}$, they interact:

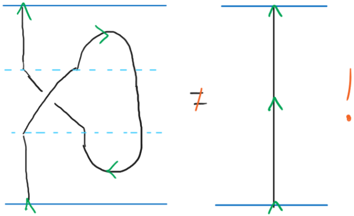


In other words, we would need to impose certain compatibility between the braidings and the duals in $\mathscr{T}$.

This is not desirable as they are independent categorical concepts!

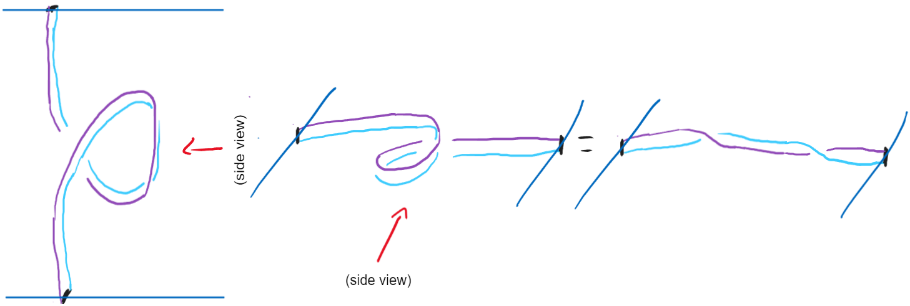
# Ribbon - geometric motivation

Solution: (similar to orientation), introduce additional geometric structure to $\mathscr{T}$ such that




This morphism then corresponds to further additional structure on the category, similar to the braiding and the cup/cap.

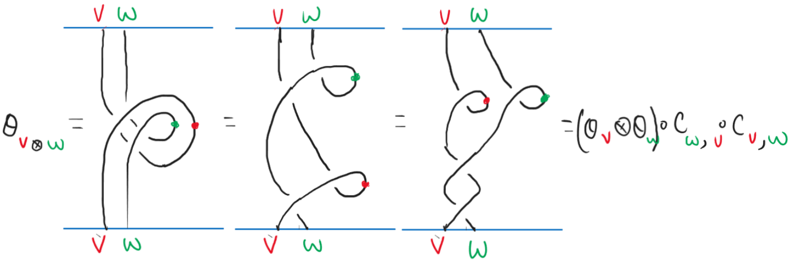
Introduce a **framing** on tangles:

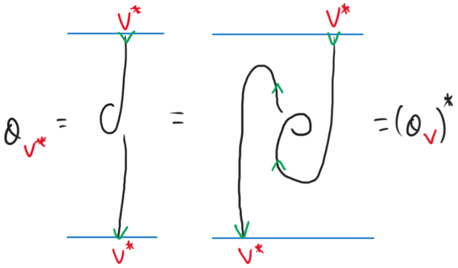



Now this morphism corresponds to a **twist** in the framing which is no longer equal to the identity.

$\mathscr{T}$ now has natural isomorphisms $\theta_V$, called the **twist**:

compatible with the braiding, and with the left duality, as follows:



## Definition

A monoidal category, braided and with **left** duals, with natural isomorphisms $\theta_V$ such that

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W) \circ c_{W,V} \circ c_{V,W}$$

and

$$\theta_{V^*} = (\theta_V)^*$$

is called a **ribbon category**.

**The category $\mathscr{T}$ of framed oriented tangles is a ribbon category.**

# On right duality

The definition of ribbon category does not include right duality.

In $\mathscr{T}$, the right dual *object* is always equal to the left dual *object*.

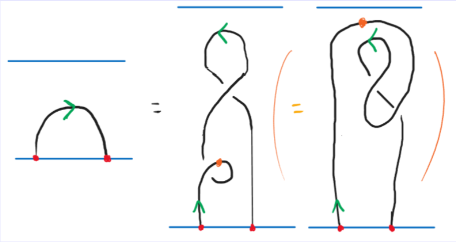
It is just that the ev and coev *morphisms* are different:
for left duality it is leftward pointing cup/cap,
for right duality it is rightward pointing cup/cap.

In general,

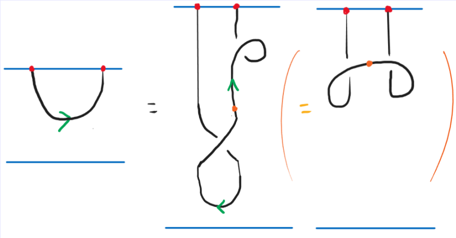
**Proposition**

*Any ribbon category has a canonical right duality (hence rigid), where* ${}^*V = V^*$ *for every object* $V$, *and* $ev'$, $coev'$ *are defined by*

$$ev'_V := ev_V \circ c_{V,V^*} \circ (\theta_V \otimes id_{V^*})$$



$$coev'_V := (id_{V^*} \otimes \theta_V) \circ c_{V,V^*} \circ coev_V .$$



# Why framing/ribbons?

## Remark

Introducing framings is a very natural thing to do in knot theory/low-dim topology. By a fundamental theorem of Lickorish-Wallace, any (closed, oriented) 3-manifold may be obtained by performing **surgery** on a framed oriented link.

This allows us to extend our story to define invariants of 3-manifolds using modular $\approx$ ribbon $+$ fusion $\approx$ ribbon $+$ finite $+$ semisimple categories.

Categorically, there is no more additional 'structure', but just algebraic 'niceness' properties.

A bit more on this at the end...

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1. Axiom and $\mathscr{T}$: how the geometry naturally gives rise to categorical axioms.
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# The main theorem 1

## Theorem

*The category $\mathscr{T}$ of framed oriented tangles is the* **universal ribbon category** *in the following sense:*
*For any ribbon category $\mathscr{C}$ and object $V$ of $\mathscr{C}$,*
*there exists a unique (strict) monoidal functor preserving braiding, duality, twist,*

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$
$$pt^{\wedge} \mapsto V$$

## Proof sketch

We have seen how to define $\mathscr{F}$ on the generators in $\mathscr{T}$ (they are precisely the braiding, duality, twist).
Then the categorical axioms satisfied by any ribbon category $\mathscr{C}$ are precisely the (minimal) set of relations satisfied by the generators in $\mathscr{T}$.
(This is a purely geometric statement in $\mathscr{T}$.)

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# Monoidal - algebraic motivation

$\mathscr{C}$ = representation category of $A$. How to make it a monoidal category?

i.e.: How to define tensor product of $A$-reps?

$V_1, V_2$ $A$-mods; $V_1 \otimes V_2$ is 'only' an $A \otimes A$-mod.

Need an algebra morphism $\Delta : A \to A \otimes A$ to pull the action back by.

Also need a representation $\mathbb{F}$ as the unit under $\otimes$, i.e. an algebra morphism $\epsilon : A \to \mathbb{F}$.

$\Delta, \epsilon$ make $A$ a bialgebra!

# Braiding - algebraic motivation

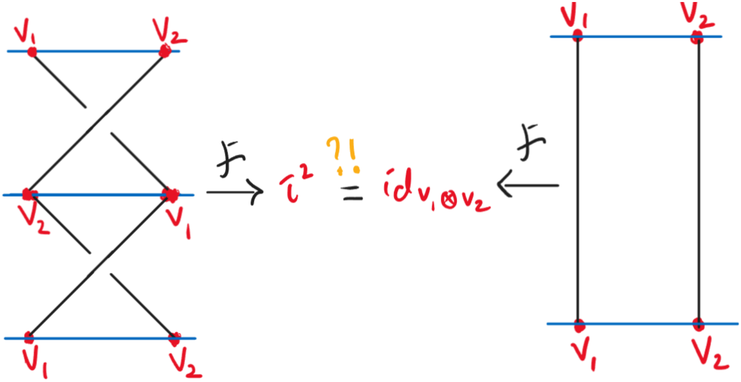
Representation category $\mathscr{C}$ braided = give nat. iso. between $V_1 \otimes V_2$ and $V_2 \otimes V_1$.

For one, there is the obvious 'flip' map $\tau : v_1 \otimes v_2 \mapsto v_2 \otimes v_1$.

This is a nat. iso. of $A$-mods only if $A$ is cocommutative.

Most common examples in representation theory are cocommutative and that is why we usually do not think much of this isomorphism.

However, if we use the flip $\tau$ (or any symm. braiding) to define $\mathscr{F}$,




i.e. pulling any two strands past each other gives the same invariant.

This invariant will not capture the geometry of the 'knottedness', and will be completely uninteresting!

*One perspective: quantum groups are the answer to the search for good examples of non-cocomm. bialgs. or nonsymmetric braidings.*

Recall $V_1 \otimes V_2$ has $A$ action via $A \xrightarrow{\Delta} A \otimes A$.

$V_2 \otimes V_1$ also has $A$ action via $A \xrightarrow{\Delta} A \otimes A$.
So we can transfer this action to $V_1 \otimes V_2$ via the flip $\tau$.

Then $V_1 \otimes V_2$ has $A$ action via $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\tau} A \otimes A$.

(If $\Delta = \tau \circ \Delta$ i.e. $A$ cocomm., then we already have our isomorphism $\tau$.)

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(If $\Delta = \tau \circ \Delta$ i.e. $A$ cocomm., then we already have our isomorphism $\tau$.)

To say that $V_1 \otimes V_2$, $V_2 \otimes V_1$ are naturally isomorphic: above two maps are conjugate in $A \otimes A$!

i.e. an invertible $R \in A \otimes A$ s.t.

$$\tau \circ \Delta = R \Delta R^{-1}.$$

The induced braiding iso. is $c_{V_1,V_2} := \tau \circ R : V_1 \otimes V_2 \to V_2 \otimes V_1$.

The braiding axiom (compatibility with $\otimes$) translates to

$$(\mathrm{id} \otimes \Delta)(R) = R^{13} R^{12}$$

**Definition**

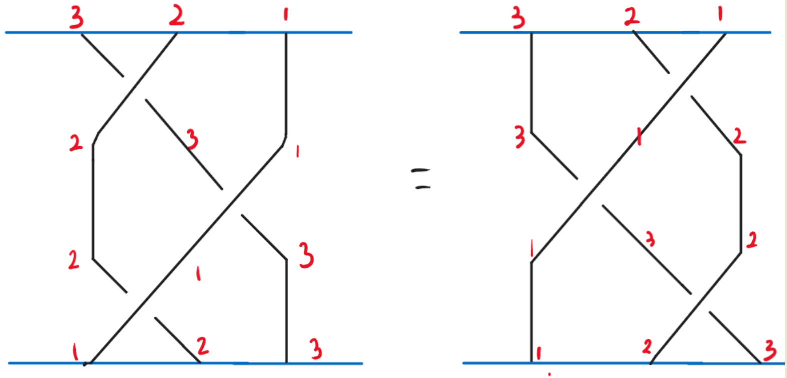
Such $(A, R)$ is called a *quasitriangular bialgebra*.

## Remark

$R$ satisfies the *quantum Yang-Baxter equation*

$$R^{23}R^{13}R^{12} = R^{12}R^{13}R^{23}$$

which just corresponds to the braid relation




(which follows from the braiding axioms).

# Duality - algebraic motivation

$V$ an $A$-mod; how to define the dual representation $V^*$?

In usual case (say group representations): need notion of 'inverse' in $A$, and define "$(a \cdot f)(v) = f(a^{-1}v)$ " for $f \in V^*, v \in V$.

So need an 'inverse' map $S : A \to A$: linear and an anti-homomorphism, and define the dual rep $V^*$ by

$$(a \cdot f)(v) := f(S(a)v)$$

for $f \in V^*, v \in V$.

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for $f \in V^*, v \in V$.

For the usual vector space evaluation and coevaluation to be $A$-equivariant:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \eta\circ\epsilon} & & \downarrow{\scriptstyle S\otimes \mathrm{id}} \\ A & \xleftarrow[\mu]{} & A \otimes A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \eta\circ\epsilon} & & \downarrow{\scriptstyle \mathrm{id}\otimes S} \\ A & \xleftarrow[\mu]{} & A \otimes A \end{array}$$

**Definition**

Such $S$ is called an *antipode* of $A$, and $(A, S)$ is called a Hopf algebra.

The repn category of a Hopf algebra has left duals, with ev and coev given by the usual vector space ev and coev.

# Ribbon - algebraic motivation

We want nat. isos. $\theta_V : V \to V$ for $V$ an $A$-mod.

This should be given by left multiplication by an invertible central element $v \in A$.

The two axioms on compatibility with braiding and left duals translate to

$$\Delta(v) = (v \otimes v)(R_{21}R)^{-1}$$

and

$$S(v) = v.$$

**Definition**

Such $(A, R, S, v)$ is called a **ribbon quasitriangular Hopf algebra**.

The upshot:

1. Recall: any ribbon category is automatically (canonically) rigid;
2. But, for $\mathscr{C}$ the ribbon representation category of a ribbon Hopf algebra $A$,
   the right duality evaluation and coevaluation is **no longer the usual vector space evaluation and coevaluation** (but is defined simply in terms of the braiding, left duality and twist).

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# The main theorem 2

**Theorem**

*If $(A, R, s, v)$ is a ribbon quasitriangular Hopf algebra, its representation category $\mathscr{C}$ of f.d. A-mods is naturally a ribbon category.*

*The converse (reconstruction theory) will also be true under certain conditions (e.g. if A is f.d.).*

**Proof sketch**

Just a matter of putting together and verifying all that we have discussed earlier.

# The million-dollar question

**Million-dollar question: Do ribbon quasitriangular Hopf algebras exist??**

Start with a well-understood representation category.

Complex semisimple Lie algebras $\mathfrak{g}$:
f.d. representations completely classified by **highest weight theory**.

To avoid excessive technicality, focus on $\mathfrak{g} = \mathfrak{sl}_2$ (building block for all other $\mathfrak{g}$).

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Complex semisimple Lie algebras $\mathfrak{g}$:
f.d. representations completely classified by **highest weight theory**.

To avoid excessive technicality, focus on $\mathfrak{g} = \mathfrak{sl}_2$ (building block for all other $\mathfrak{g}$).

Problem: $\mathfrak{g}$-reps are too well-behaved.

$V \otimes W \cong W \otimes V$ via the flip $\tau$; $\mathbf{U}(\mathfrak{g})$ is cocommutative!

**Idea:** Deform $\mathbf{U}(\mathfrak{g})$ by a parameter $q$; hopefully, the reps can still be classified analogously, but the algebraic structure is more interesting.

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### Recall

$\mathbf{U}(\mathfrak{sl}_2(\mathbb{C}))$ is the $\mathbb{C}$-algebra generated by $E, F, H$ modulo

$$EF - FE = H$$

$$EH - HE = -2E \iff EH = (H-2)E$$

$$FH - HF = 2F \iff FH = (H+2)F$$

### Definition

$\mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ is the $\mathbb{C}(q)$-algebra generated by $E, F, K, K^{-1}$, modulo

$$KK^{-1} = K^{-1}K = 1$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$EK = q^{-2}KE$$

$$FK = q^2 KF$$

$q$ here is an indeterminate sometimes called the quantum parameter.

## Recall

$\mathbf{U}(\mathfrak{sl}_2(\mathbb{C}))$ is a bialgebra via:

$$\Delta(E) = E \otimes 1 + 1 \otimes E$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H$$
$$\epsilon(E) = \epsilon(F) = \epsilon(H) = 0$$

## Proposition

$\mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ *is a bialgebra via:*

$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$
$$\Delta(K) = K \otimes K$$
$$\epsilon(E) = \epsilon(F) = 0 \quad \epsilon(K) = 1$$

## Remark

Why the name quantum *group* for $\mathbb{C}(q)$-algebras?
Philosophically speaking, we are trying to obtain a quantization of the *group* $SL_2(\mathbb{C})$ by deforming its (infinitesimal) function space $\mathbf{U}(\mathfrak{sl}_2(\mathbb{C}))$.

For $G$ an algebraic group: just as

$$G\text{-mods} \leftrightsquigarrow \mathbf{U}(\mathfrak{g})\text{-mods} \leftrightsquigarrow \mathscr{O}(G)\text{-comods},$$

we have

$$\mathbf{U}_q(\mathfrak{g})\text{-mods} \leftrightsquigarrow \mathscr{O}_q(G)\text{-comods},$$

$\mathscr{O}_q(G)$ is the *quantum function algebra*.

## Recall

Every f.d. $\mathbf{U}(\mathfrak{sl}_2(\mathbb{C}))$-mod is semisimple.
The f.d. irreducible $\mathbf{U}(\mathfrak{sl}_2(\mathbb{C}))$-mods are precisely $V_n$ of dimension $n$ and highest weight $(n-1)$, for all positive integers $n$.

## Proposition

*Every f.d. $\mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$-mod is semisimple.*
*The f.d. irreducible $\mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$-mods are precisely $V_{n,\epsilon}$ of dimension $n$ and highest weight $\epsilon q^{n-1}$, for all positive integers $n$ and sign $\epsilon = \pm 1$.*

## Theorem

$\mathbf{U}_q := \mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ *is a ribbon quasitriangular Hopf algebra with braiding (R-matrix), antipode (S), twist (v) given by*

$$R \stackrel{!}{=} e^{h(H\otimes H)/4} \sum_{n\geq 0} \frac{(q-q^{-1})^n}{[n]!} q^{n(n-1)/2} (E^n \otimes F^n)$$

$$S(E) = -EK^{-1} \quad S(F) = -KF \quad S(K) = K^{-1}$$

$$v \stackrel{!}{=} e^{-hH^2/4} \sum_{n\geq 0} \frac{(q^{-1}-q)^n}{[n]!} q^{n(3n+1)/2} F^n K^{-n-1} E^n$$

*with* $q = e^h, K = e^{hH/2}$; $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ *are called quantum integers.*

## Remark

The statement above is not strictly correct!

$$R \overset{!}{=} e^{h(H\otimes H)/4} \sum_{n\geq 0} \frac{(q-q^{-1})^n}{[n]!} q^{n(n-1)/2} (E^n \otimes F^n)$$

$$v \overset{!}{=} e^{-hH^2/4} \sum_{n\geq 0} \frac{(q^{-1}-q)^n}{[n]!} q^{n(3n+1)/2} F^n K^{-n-1} E^n$$

## Remark

$R, v$ contain infinite sums and hence do not define genuine elements of $\mathbf{U}_q$.

The solution, naturally, is to work in a completion of $\mathbf{U}_q$ (over $\mathbb{C}[[h]]$ with the $h$-adic topology).

However, this is not a show-stopper because all we need are the induced morphisms on $\mathbf{U}_q$-mods. These morphisms can still be defined because e.g. $E, F$ will act nilpotently on all of them.

This is morally comparable to ignoring convergence issues and working formally with infinite sums in analysis.

# The main theorems

## Recall

The category $\mathscr{T}$ of framed oriented tangles is the **universal ribbon category** in the following sense:
For any ribbon category $\mathscr{C}$ and object $V$ of $\mathscr{C}$,
there exists a unique (strict) monoidal functor preserving braiding, duality, twist,

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$
$$\mathrm{pt}^{\wedge} \mapsto V$$

## Recall

If $(A, R, s, v)$ is a ribbon quasitriangular Hopf algebra, its representation category $\mathscr{C}$ of f.d. $A$-mods is naturally a ribbon category.

The converse (reconstruction theory) will also be true under certain conditions (e.g. if $A$ is f.d.).

# Reshetikhin-Turaev invariant

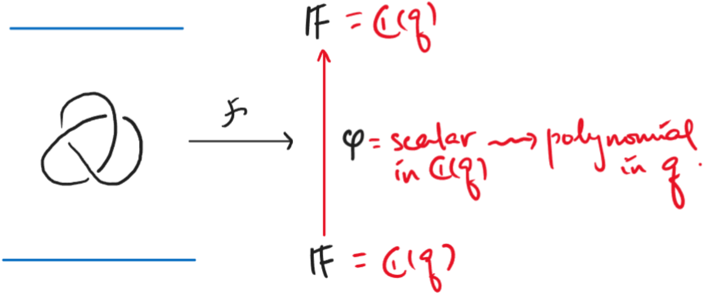
**Theorem**

*Applying the main theorem to obtain a functor*

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$

$$pt^{\wedge} \mapsto V_{2,+}$$

*for the 'standard' 2-dim rep* $V_{2,+}$ *of* $\mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$*,*
*one recovers (modulo constants/signs etc) the* **Jones polynomial***.*



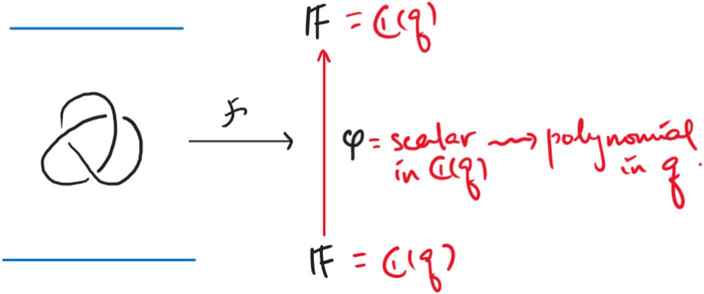
## Theorem

*Applying the main theorem to obtain a functor*

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$

$$pt^{\wedge} \mapsto V_{n,+}$$

*for the n-dim rep $V_{n,+}$ of $\mathbf{U}_q(\mathfrak{sl}_2(\mathbb{C}))$,*
*one recovers (modulo constants/signs etc) the* **coloured Jones polynomial**.



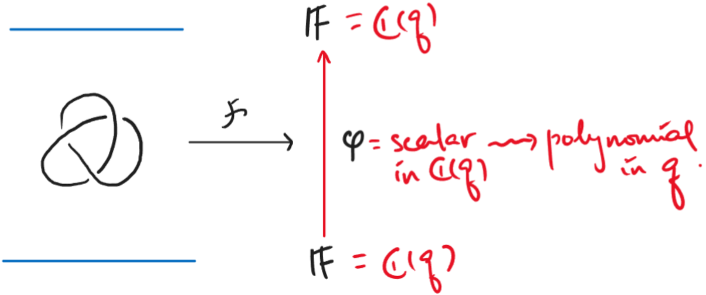
### Theorem

*Applying the main theorem to obtain a functor*

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$

$$pt^{\wedge} \mapsto V_{n,+}$$

*for the 'standard' n-dim rep* $V_{n,+}$ *of* $\mathbf{U}_q(\mathfrak{sl}_n(\mathbb{C}))$*,*
*one recovers (modulo constants/signs etc) the* **HOMFLYPT polynomial***.*



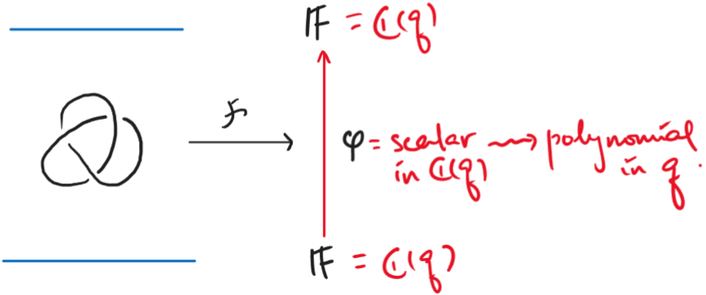
## Theorem

*Applying the main theorem to obtain a functor*

$$\mathscr{F} : \mathscr{T} \to \mathscr{C}$$

$$pt^{\wedge} \mapsto V$$

*for $V$ 'standard' reps of $\mathbf{U}_q(\mathfrak{g})$ for $\mathfrak{g}$ other classical Lie algebras $(\mathfrak{sp}, \mathfrak{so})$, one recovers (modulo constants/signs etc) the* **Kauffman polynomial**.



We have seen an alternative way of producing similar knot invariants via the Temperley-Lieb algebra/category.

Very natural question: is there a relation?

We have seen an alternative way of producing similar knot invariants via the Temperley-Lieb algebra/category.

Very natural question: is there a relation?

Somewhat surprising answer: yes, via **quantum Schur-Weyl duality**!

**Recall ('Usual' Schur-Weyl)**

$V$ standard $n$-dim representation of $\mathbf{U}(\mathfrak{sl}_n(\mathbb{C}))$.
$V^{\otimes m}$ has both $\mathbf{U}(\mathfrak{sl}_n(\mathbb{C}))$-action (via tensor product) and $S_m$-action (via permuting factors), so we have the maps

$$\mathbf{U}(\mathfrak{sl}_n(\mathbb{C})) \to \mathrm{End}(V^{\otimes m}) \leftarrow \mathbb{C}[S_m]$$

whose images are centralisers of each other.

When $n = 2$, the image of $\mathbb{C}[S_m]$ is precisely the Temperley-Lieb algebra specialised at $\delta = -2$, i.e. we have an isom.

$$\mathrm{TL}_{m,-2} \cong \mathrm{End}_{\mathbf{U}(\mathfrak{sl}_2)}(V^{\otimes m})$$

## Theorem (Quantum Schur-Weyl)

*$V$ standard n-dim representation of $\mathbf{U}_q(\mathfrak{sl}_n(\mathbb{C}))$.*
*$V^{\otimes m}$ has both $\mathbf{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-action (via tensor product) and $Br_m$-action (by permuting factors using the braiding/R-matrix), so we have the maps*

$$\mathbf{U}_q(\mathfrak{sl}_n(\mathbb{C})) \to End(V^{\otimes m}) \leftarrow \mathbb{C}[Br_m]$$

*whose images are centralisers of each other.*

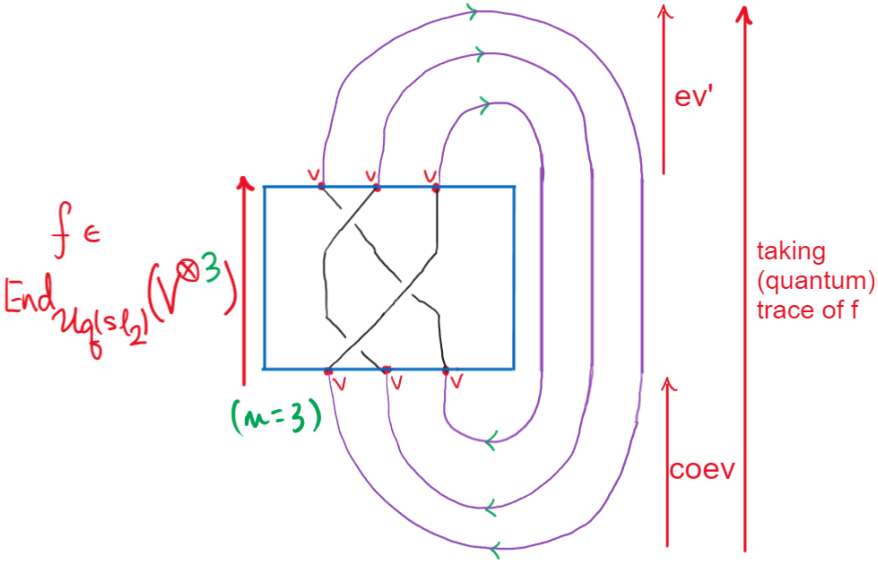
*When $n = 2$, the image of $\mathbb{C}[Br_m]$ is precisely the Temperley-Lieb algebra specialised at $\delta = -[2] = -(q + q^{-1})$, i.e. we have an isom.*

$$TL_{m,-[2]} \cong End_{\mathbf{U}_q(\mathfrak{sl}_2)}(V^{\otimes m})$$

## Remark

Alternative motivation: the Temperley-Lieb algebra arises in a certain graphical calculus used to study the repn structure of $\mathbf{U}_q(\mathfrak{sl}_2)$.

Temperley-Lieb invariant: start with an $m$-strand braid, map it to an element of $\mathrm{TL}_{m,-(q+q^{-1})} \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes m})$, then take 'trace'.



## Remark (What next?)

There are slightly more systematic ways of producing quantum groups: 'Drinfeld's quantum double' produces from any (f.d.) Hopf algebra an (f.d.) quasitriangular Hopf algebra. In this way we can obtain e.g. a (f.d.) quotient of $\mathbf{U}_q$ (for $q$ a root of unity).

- The repn theory of $\mathbf{U}_q$ for $q$ a root of unity is akin to the repn theory of $\mathfrak{g}$ in +ve characteristic;
- The f.d. quotient is akin to taking the restricted universal enveloping algebra in +ve characteristic;
- The corresponding representation category has finitely many simples.
- After suitable semisimplification process ('taking quotient by negligible modules'), we obtain a modular ($\approx$ ribbon + finite + semisimple) category used to produce the invariants of 3-manifolds alluded to earlier.
- These invariants coincide with certain others defined via 3D TQFTs; in fact, from any modular category one can produce a 3D TQFT.
- The story of *quantum invariants* continues... ...

Thank you!