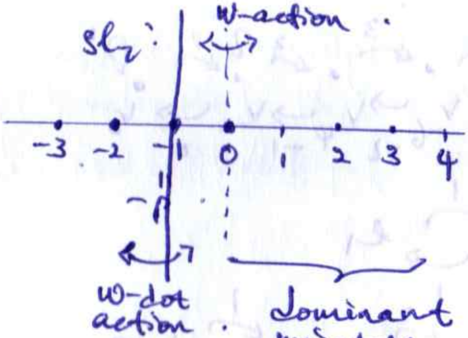
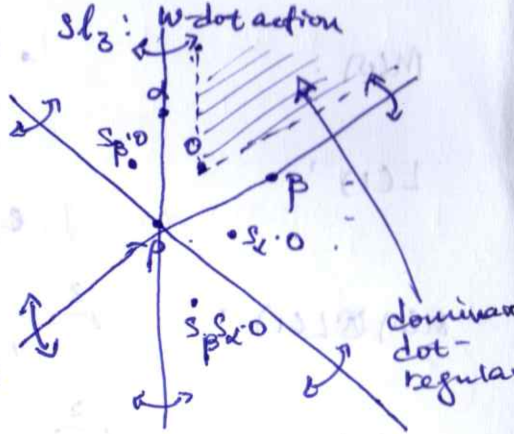
Recall: $\Lambda_r \subset \Lambda$

root lattice ; (integral) weight lattice

Fix Δ, $\rho = \frac{1}{2}$ sum of +ve roots.





Focus on integral weights (Λ).

Recall: block decomposition.

$$\mathcal{O} = \bigoplus_{\chi_\lambda} \mathcal{O}_{\chi_\lambda} := \bigoplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\lambda \qquad \mathcal{O}_{int} = \bigoplus_{\lambda \in \Lambda/(W,\cdot)} \mathcal{O}_\lambda \leftarrow \text{true block decomposit}^{\text{n}}$$

$[M(w\cdot\lambda)]$ is $\mathbb{Z}$-basis for $K(\mathcal{O}_{\chi_\lambda})$ $\Rightarrow$ dominant λ have isomorphic $K(\mathcal{O}_\lambda)$!

$[L(w\cdot\lambda)]$

$[P(w\cdot\lambda)]$

Q1: How are $\mathcal{O}_\lambda$ related for (dominant) λ? Can we lift the isomorphism (to $\mathcal{O}_0$) of $K(\mathcal{O}_\lambda)$ to equivalence of categories?

$\mathcal{O}_0$ ← principal block.

Example in sl_2 case. How to bring $M(0)$ to $M(1)$? ← $\underbrace{M(0) \text{ to } M(1)}_{\text{increase weight}}$ → tensor.

Consider $M(0) \otimes L(1)$.

$M(0)$:

$$\begin{tikzcd} \cdot & v_{-6} \arrow[l, bend right, "4"'] \arrow[r, bend right, "-2"'] & v_{-4} \arrow[l, bend right, "3"'] \arrow[r, bend right, "-1"'] & v_{-2} \arrow[l, bend right, "2"'] \arrow[r, bend right, "0"'] & v_0 \arrow[l, bend right, "1"'] \end{tikzcd}$$

$L(1)$:

$$\begin{tikzcd} e_{-1} \arrow[r, bend right] & e_1 \arrow[l, bend right, "1"'] \end{tikzcd}$$

$M(0) \otimes L(1)$:

$$\begin{tikzcd} \cdot & v_{-4}\otimes e_1 \arrow[l, bend right, "3"'] \arrow[r, bend right, "1"'] & v_{-2}\otimes e_1 \arrow[l, bend right, "2"'] \arrow[r, bend right, "0"'] \arrow[dl, "1"] & v_0 \otimes e_1 \arrow[l, bend right, "1"'] \arrow[dl, "1"] \\ \cdot \arrow[r, bend right] & v_{-2}\otimes e_{-1} \arrow[l, bend right, "2"'] \arrow[ur, "2"] \arrow[r, bend right, "0"'] & v_0 \otimes e_{-1} \arrow[l, bend right, "1"'] \arrow[ur, "1"] & \end{tikzcd}$$

$-1 \qquad -3 \qquad -1 \qquad 1$

(gen. by $v_0 \otimes e_1$, $v_0 \otimes e_{-1}$).

$\cup$

$M(1)$ (submod. gen by highest weight $v_0 \otimes e_1$).

$$\begin{tikzcd} \cdot & {\begin{array}{c} v_{-4}\otimes e_1 \\ + \\ v_{-2}\otimes e_{-1} \end{array}} \arrow[l, bend right] \arrow[r, bend right, "0"'] & {\begin{array}{c} v_{-2}\otimes e_1 \\ + \\ v_0 \otimes e_{-1} \end{array}} \arrow[l, bend right, "2"'] \arrow[r, bend right, "1"'] & v_0 \otimes e_1 \arrow[l, bend right, "1"'] \end{tikzcd}$$

$\dfrac{M(0)\otimes L(1)}{M(1)} \cong M(-1)$:

$$\begin{tikzcd} \cdot \arrow[r, bend right, "-2"'] & \overline{v_{-2}\otimes e_{-1}} \arrow[l, bend right, "2"'] \arrow[r, bend right, "-1"'] & \overline{v_0 \otimes e_{-1}} \arrow[l, bend right, "1"'] \end{tikzcd}$$

$L(1)$ has weights $1, -1$ corresponding to e_1, e_{-1}. $M(0) \otimes L(1)$ has filtration $M(1), M(-1)$ corresponding to $v_0 \otimes e_1$, $v_0 \otimes e_{-1}$.

Thm 1. V f-d w/ basis of weight vectors $v_1, \dots, v_n$.

weights $\mu_1 \le \dots \le \mu_n$.

Then $V \otimes M(\lambda)$ has std. filtration

$$0 \subset M_n \subset M_{n-1} \subset \dots \subset M_i \subset \dots \subset M_1 = V \otimes M(\lambda) .$$

gen. by $v_i \otimes v^+, \dots, v_n \otimes v^+$. ($v^+$ highest weight in $M(\lambda)$)

$M_i / M_{i+1} \cong M(\lambda + \mu_i)$.

Pf. ① Show $M_1 = V \otimes M(\lambda)$.

M_1 gen. by $v \otimes v^+$ for $v \in V$.

By PBW, suff. show $v \otimes uv^+ \in M_1$ $\forall$ PBW monomial $u \in U(\mathfrak{g})$.

But $\forall x \in \mathfrak{g}$, $v \otimes x(u'v^+) = x(v \otimes u'v^+) - xv \otimes u'v^+$.

Inductively, $\in M_1$.

② M_i / M_{i+1} gen. by $v_i \otimes v^+$ of weight $\lambda + \mu_i$.

$\mathfrak{n}$ kills $v_i \otimes v^+$ $\because$ $\mathfrak{n}$ kills v^+

and $\mathfrak{n}$ sends v_i to higher weight

& weight ordering $\mu_i \le \mu_{i+1} \le \dots$

so sends $v_i \otimes v^+$ to M_{i+1}.

So have surjections $M(\lambda + \mu_i) \twoheadrightarrow M_i / M_{i+1}$ $\forall i$.

③.

③ Intuitively, only thing left to do is "dimension count",
i.e. compare formal characters:

$$\begin{aligned} ch\, V \otimes M(\lambda) &= ch\, V * ch\, M(\lambda) \\ &= \sum e^{\mu_i} * e^{\lambda} \prod_{\alpha \in \Phi^-} \frac{1}{1-e^{\alpha}} \\ &= \sum e^{\lambda+\mu_i} \prod_{\alpha \in \Phi^-} \frac{1}{1-e^{\alpha}} \\ &= \sum ch\, M(\lambda+\mu_i) \end{aligned}$$

$\square$

Alternate proof: Use tensor identity (abstract nonsense)
then exactness of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \cdot$ ← need PBW to say $U(\mathfrak{g})$ is free over $U(\mathfrak{b})$

More concrete construction above.

Recall: $L(\lambda)$ f.d. $\Leftrightarrow$ λ dominant; every f.d. mod is $\oplus$ of $L(\lambda)$.

Together w Thm 1, motivates the following:

↓

tells us we can increase weight additively by tensoring w f.d. module of the desired weight increase

Def: We are going to define functors
$$T_\lambda^\mu : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\mu .$$
by tensoring w/ f.d. module dependent on $\mu - \lambda$.
Let ν = unique dominant weight in W-orbit of $\mu - \lambda$
(recall here weights of $L(\cdot)$ are closed under W-~~action~~ taking conjugate)
Then: $T_\lambda^\mu := pr_\mu \circ \left(L(\nu) \otimes \cdot \right) \circ i_\lambda$.
Recall tensoring w/ f.d. is exact ($\mathcal{O}$ closed under, & exact for vector spaces)
"translation" functor.
↳ think of 'sliding' along Euclidean plane in sl_3
Rmk: Easy to see it preserves projectives.

Natural qn: What is T_μ^λ?
Expect dual. $L(\nu)^*$ clearly also simple (submod $\longleftrightarrow$ quotient) and f.d.
Weights are $-$ve that of $L(\nu)$.
Recall: $\dim L(\nu)_\mu = \dim L(\nu)_{w\mu}$ $\forall$ w, μ.
~~So~~ Lowest weight of ~~$L(\lambda)$~~ $L(\nu)$ therefore $w_0 \nu$
$\Rightarrow L(\nu)^* \cong L(-w_0 \nu)$ dominant $\because$ f.d.

$\qquad -w_0 \nu = -w_0 w(\mu - \lambda) = w_0 w(\lambda - \mu)$ $\exists$ w

so W-conjugate to $\lambda - \mu$.

Upshot: $T_\mu^\lambda = pr_\lambda \circ (L(\nu)^* \otimes \cdot) \circ i_\mu$.

Prop: T_λ^μ, T_μ^λ biadjoint.

Actually, Jinfeng showed natural isomorphism
$$\mathrm{Hom}_{\mathcal{O}}(V \otimes M, N) \cong \mathrm{Hom}_{\mathcal{O}}(M, V^* \otimes N)$$
(proof is just really checking compatibility w.r.t. $U(\mathfrak{g})$) $\square$

Corr. of Thm. 1: $T_\lambda^\mu(M(w\cdot\lambda))$ has filtrat^n w/ sub quotients
$$M(w\cdot\lambda+\nu')$$
where ν' is weight of $L(\nu)$,
$$w\cdot\lambda+\nu' \in W\cdot\mu.$$

Now to study effect of T_λ^μ on Verma, largely reduced to weight problem.
Keeping in mind also example earlier for sl_2 $M(0)\otimes L(1)$, have:

Prop: μ, $\lambda-\mu$ (and λ) dominant integral. Then:

① $T_\mu^\lambda(M(w\cdot\mu)) \cong M(w\cdot\lambda)$, (tensoring w $L(\lambda-\mu)$)

② $T_\lambda^\mu(M(w\cdot\lambda)) \cong M(w\cdot\mu)$ (tensoring w $L(w_0(\mu-\lambda))$
$L(\lambda-\mu)^*$
w lowest weight $\mu-\lambda$.)

⑥

Pf: (1) Intuitively, $\lambda-\mu$ already maximum in $L(\lambda-\mu)$, so that should be the "only way" to go from μ to λ. (cf. sl_2 example: $0 \to 1$ via $L(1)$)
Formally, will have to be some sort of inequality bounding.

If $\exists$ weight ν' of $L(\lambda-\mu)$ s.t.

$$w\cdot\mu + \nu' = x\cdot\lambda$$

$$\Leftrightarrow x^{-1}w(\mu+\rho) + \underbrace{x^{-1}\nu'}_{\text{max is } \lambda-\mu} = \lambda+\rho.$$

(upper bounded by $\lambda-\mu$)

$$\Rightarrow x^{-1}w(\mu+\rho) + (\lambda-\mu) \geq \lambda+\rho$$

$$\Rightarrow x^{-1}w\cdot\mu \geq \mu$$

But μ dominant (*)

$$\Rightarrow x^{-1}w\cdot\mu = \mu \Rightarrow \underline{\text{force equality throughout}}$$

$$\Rightarrow x = w \quad (\nu' = x(\lambda-\mu) = w(\lambda-\mu) = w\cdot\lambda - w\cdot\mu)$$ ξ_1: conjugate to $\lambda-\mu$, each multiplicity 1

(2) Same idea but lower bound by $\mu-\lambda$. (cf. sl_2 example: $1 \to 0$ via $L(1)$ lowest weight -1)

$$w\cdot\lambda + \nu' = x\cdot\mu$$

$$\Rightarrow \lambda+\rho+(\mu-\lambda) \leq \lambda+\rho+w^{-1}\nu' = w^{-1}x(\mu+\rho)$$

$$\Rightarrow \mu \leq w^{-1}x\cdot\mu$$

But μ dominant (*)

$\Rightarrow \mu = w^{-1}{}_x \cdot \mu \Rightarrow$ force equality throughout

$\Rightarrow \nu' = w(\mu - \lambda)$, multiplicity 1. $\square$.

<u>Rmk</u>: As indicated in (*), we need actually only μ to be maximal in dot-orbit.

Then (1) becomes $T_\mu^\lambda(M(w\cdot\mu))$ has filtrat^n ~~=~~ as $M(ww'\cdot\lambda)$ where w' runs over dot-stabilizer of μ.

(2) is still the same.

Intuitively, when "going up" from μ to λ, may go up to weights W-linked to λ (-1) ~~0 to 1~~ -1 to 0.

But when "going down" to μ, should be unique (-1 only)

We will need this later. (0 to -1) Rmk 2: If $w = id$, observe $M(\lambda) = P(\lambda)$ in (1). This is not a coincidence.

Summary: $T_\lambda^\mu, T_\mu^\lambda$ are exact, adjoint functors which send (λ, μ dominant) $M(w\cdot\lambda)$ to $M(w\cdot\mu)$ and vice versa, i.e. they induce inverse isomorphisms on the Grothendieck groups.

<u>Thm 2</u>: $T_\lambda^\mu, T_\mu^\lambda$ are inverse equivalences between $\mathcal{O}_\lambda, \mathcal{O}_\mu$.

<u>Pf</u>: Though abstract nonsense, but proof is actually quite interesting.
this/thingy way
makes it as

(8)

I: Since adj. adjoint, suffice show unit (counit similar) is natural isomorphism, i.e. $\forall\, M \in \mathcal{O}_\lambda$,

$$M \xrightarrow{\phi} T_\mu^\lambda T_\lambda^\mu M \text{ is iso.} \qquad (\text{Else } [\operatorname{coker}\phi] = 0)$$

Actually $\because [M] = [T_\mu^\lambda T_\lambda^\mu M]$, suff. show injective $\wedge$ ~~(∵ $\mathcal{O}_\lambda$ Artinian!)~~.

Natural to consider adjointness property:

$$\begin{tikzcd} T_\lambda^\mu M \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & T_\lambda^\mu T_\mu^\lambda T_\lambda^\mu M \arrow[r] & T_\lambda^\mu M \end{tikzcd}$$

So $T_\lambda^\mu M \hookrightarrow T_\lambda^\mu T_\mu^\lambda T_\lambda^\mu M$ injective.

But now T_λ^μ exact. So if $M \xrightarrow{\phi} T_\mu^\lambda T_\lambda^\mu M$ not injective,

short exact $0 \to \ker\phi \hookrightarrow M \xrightarrow{\phi} \operatorname{im}\phi \subset T_\mu^\lambda T_\lambda^\mu M \to 0$

gives short exact $0 \to T_\lambda^\mu(\ker\phi) \hookrightarrow T_\lambda^\mu M \twoheadrightarrow \underbrace{T_\lambda^\mu T_\mu^\lambda T_\lambda^\mu M}_{T_\lambda^\mu(\operatorname{im}\phi)} \to 0$

So $T_\lambda^\mu(\ker\phi) = 0$

But T_λ^μ is iso. on Grothendieck groups

$\Rightarrow [\ker\phi] = 0 \Rightarrow \ker\phi = 0$. $\square$.

Cor: All $\mathcal{O}_\lambda$ (λ dominant) equiv. to $\mathcal{O}_0$. (May reduce to study of $\mathcal{O}_0$.)

Q^n: How to study structure of $\mathcal{O}_0$? (Write $M_w := M(w\cdot 0)$).

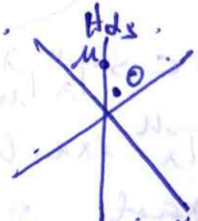
$\longrightarrow$ Compose translat^n functors to study $\mathcal{O}_0$ by means of endofunctors.

Choice of endofunctors on $\mathcal{O}_0$: $\mathcal{O}_0 \to \mathcal{O}_\mu \to \mathcal{O}_0$.

By what we know, if μ dominant, $T_\mu^0 \circ T_0^\mu$ is useless ~~(id on Grothendieck group)~~
(nat. iso. to id)

Next best choice: "one level down" for μ; still in ρ-shifted dominant chamber,
μ has dot-stabiliser $\{id, s\}$ for simple reflect^n s. i.e. μ lies on one root hyperplane (say H_{α_s})

Prop. only tells us what happens when $\lambda - \mu$ dominant.
So choose ν "suff. large" dominant s.t. $\nu - \mu$ also dominant.



Def: $\Theta_s := \left("T_\mu^0" \circ "T_0^\mu" \right)$
$:= T_\nu^0 \circ T_\mu^\nu \circ T_\nu^\mu \circ T_0^\nu$ ("wall-crossing"/"wall-bouncing" functors)

Look at effect of Θ_s on $\mathbb{Z}$-basis $[M_w]$ of $K(\mathcal{O}_0) \cong \mathbb{Z}[W]$.

$$[M_w] \xmapsto{T_0^\nu} [M(w\cdot\nu)] \xmapsto{T_\nu^\mu} [M(w\cdot\mu)] \xmapsto{T_\mu^\nu} [M(w\cdot\nu)] + [M(ws\cdot\nu)]$$ go to pg (12)

So Θ_s acts as (right) multiplicat^n by $(1+s)$! $\xmapsto{T_\nu^0} [M_w] + [M_{ws}]$.

(thus, as a first step, categorifies $\mathbb{Z}[W]$ equipped w/ right mult by $(1+s)$)

Q^{4}: What is significance of $(1+s)$? First, $(1+s)$ generates $\mathbb{Z}[W]$

Magic: $[P_s] = [M_{id}] + [M_s]$!

(BGG reciprocity: $(P_s : M_x) = [M_x : L_s]$; so x only id or s, and then both multiplicity 1)

So θ_s sends $[M_{id}] \longmapsto [P_s]$ { goto pg (12)

Now $[P_w]$ form a basis of $\mathbb{Z}[W]$ (sometimes denoted C_w) ← Kazhdan-Lusztig basis

→ Expect to have functors $\theta_w : [M_{id}] \longmapsto [P_w]$

Then together w $\oplus$ and $\circ$, these functors categorify right regular representation of $\mathbb{Z}[W]$ (right module action on itself).

In fact: these are all the projective endo functors on $\mathcal{O}_0$ (direct summand of tensor, of which translation is special case)

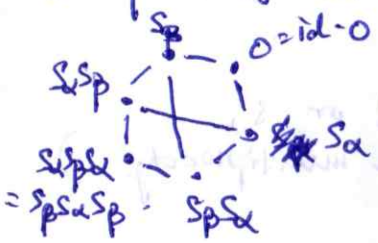
and the unique indecomposables are the $\theta_w : [M_{id}] \longmapsto [P_w]$ corresponding to the basis $[P_w] = C_w$

Further: each θ_w acts as multiplication by nonnegative linear combi. of $y \leq w$ (Bruhat)

{ if ~~θ_w~~ w has reduced exp. $s_1 \cdots s_k$

then θ_w ~~has~~ is direct summand of $\theta_{s_k} \circ \cdots \circ \theta_{s_1}$.

Example: sl_3. All composition factor multiplicities $= 1$.



Θ_{id}

Θ_{s_α} acts as $[P_\alpha] = [M_{id}] + [M_\alpha]$

$[\Theta_\beta] =$ $[P_\beta] = [M_{id}] + [M_\beta]$ ① $\Theta_\alpha \circ \Theta_\alpha = \Theta_\alpha \oplus \Theta_\alpha$.

$\Theta_{\beta\alpha} =$ $[P_{\beta\alpha}] = [M_{\beta\alpha}] + [M_\beta] + [M_\alpha] + [M_{id}]$

② $\Theta_\alpha \circ \Theta_\beta \cong \Theta_{\beta\alpha}$.

$\Theta_{\alpha\beta\alpha} = [M_{id}] + [M_\alpha] + [M_\beta] + [M_{\alpha\beta}] + [M_{\beta\alpha}] + [M_{\alpha\beta\alpha}]$
$= [P_{\alpha\beta\alpha}]$

③ $\Theta_\alpha \circ \Theta_\beta \circ \Theta_\alpha = \Theta_{\alpha\beta\alpha} \oplus \Theta_\alpha$.

①, ②, ③ are defining relations for $\mathbb{Z}[W]$ generated by $(1+s)$

Kazhdan-Lusztig: $M:L = P:M =$ relat^n between $\{C_w\}$ and std. basis $\{w\}$ of $\mathbb{Z}[W]$.

$T_\mu^\nu M_{id} = P \xrightarrow{\pi} M_s$ (lowest weight is last in filtrat^n, occurs as quotient)

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[dr, "\psi"] & M_s \\ & P_s \arrow[u, "\phi\ (\text{essential})"'] \\ P_s \arrow[uu, "\psi'", "\text{splitting}"'] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

$\Rightarrow P_s$ is $\oplus$ summand of P.

More generally, $T_\mu^\lambda(M(\mu)) \cong P(w_0' \cdot \lambda)$ where w_0' is longest ele. in dot-stab. of μ.
E.g. if $\mu = -\rho$, $\mapsto P(w_0 \cdot \lambda)$, multiplicities all 1.